Stability of maps to projective spaces, with applications to the slope of fibred surfaces

Tesi di dottorato di

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A mio padre

E andando nel sole che abbaglia
sentire con triste meraviglia
com'è tutta la vita e il suo travaglio
in questo seguitare una muraglia
che ha in cima cocci aguzzi di bottiglia.

Eugenio Montale, Ossi di seppia.

Contents

Introduction

From a general point of view, this thesis deals with some aspects of the classification of complex algebraic varieties. In particular we are interested in the *geography* of varieties, which is, roughly speaking, the study of the ranges in which their numerical invariants can vary. More precisely, geography studies the connection between the geometric properties of a variety and the inequalities (or equalities) satisfied by its invariants.

Let us consider the case of complex algebraic surfaces. If S is a minimal surface of general type, its invariants K_S^2 and $\chi_S = \chi(\mathcal{O}_S)$ satisfy the following inequalities (cf. [BHPdV04], VII. 8).

$$K_S^2 \geq 1, \quad \chi_S \geq 1, \quad K_S^2 \leq 9\chi_S \text{ (Miyaoka-Yau)}, \quad K_S^2 \geq 2\chi_S - 6 \text{ (M. Noether)}.$$

These inequalities define a (non-compact) region R in the plane. It has been proved that given any admissible pair (n, m) in this region there exist a minimal surface with $K_S^2 = n$ and $\chi_S = m$. This is a first example of a geographical problem. Another typical issue is to understand properties of the surfaces having invariants in a particular subregion of R. For instance, it is still an open question wether there are simply connected surfaces with invariants lying in the line $K_S^2 = 9\chi_S$.

We study geographical problems from a relative point of view; the objects of our study are indeed *fibred varieties*, more precisely flat proper morphisms $f\colon X \longrightarrow T$ of complex varieties. These objects are endowed with naturally associated invariants, which are the relative version of the invariants of X.

In particular we will deal with the case of *fibred surfaces*. A fibred surface is a proper surjective morphism with connected fibres from a smooth projective surface S to a smooth complete curve B. The existence, or non-existence of a fibration is a very important topological property of surfaces of general type. For instance, many of the examples of surfaces with given invariants are constructed as fibrations of genus 2. Although this is often the case -for instance if $g, b \geq 2$- we shall not suppose that the surface S is of general type.

Given a fibred surface $f\colon S \longrightarrow B$, let g be the genus of a general fibre, and b be the genus of the base B. The invariants we are mainly concerned with are the self-intersection of the relative canonical sheaf $\omega_f = \omega_S \otimes f^*\omega_B^{-1}$ and the degree of its pushforward $f_*\omega_f$:

$$(\omega_f \cdot \omega_f) = K_S^2 - 8(g-1)(b-1), \quad f_*\omega_f = \chi_S - (g-1)(b-1).$$

When the fibration is non-locally trivial, we shall call *slope* $s(f)$ the ratio between these two relative invariants. The geographical problem that we consider in this work is to find lower bounds for the slope.

Three main techniques have been used to prove lower bounds for the slope of fibred surfaces. The most widely used is the method introduced by Xiao in [Xia87a], which exploites the Harder-Narashiman filtration of the pushforwards of line bundles on S. This method has been fruitfully used and generalised subsequently. More recently, Moriwaki in a series of papers (see for instance [Mor96] and [Mor02]), developed an argument which relies on an application of Bogomolov's instability theorem. Finally, we mention the method of relative hyperquadrics (see [Kon93]and [Bar]), which is more useful in the study of the lowest cases of invariants.

In [CH88] Cornalba and Harris introduced a fourth method, different from the ones cited above, which is based on Geometric Invariant Theory (GIT). As their interest was the enumerative geometry of the moduli space of stable curves, the two authors only dealt with the slope of semi-stable fibrations, i.e. fibred surfaces whose fibres are all moduli semi-stable curves. The main aim of this thesis is to generalise this technique, and apply it to investigate the slope of fibred surfaces.

This thesis is divided in two parts, corresponding to the two chapters. The main result of the first one is a generalisation of the Cornalba-Harris theorem. This provides in particular a method for studying the numerical invariants of a fibration $f\colon X \longrightarrow T$ when T is 1-dimensional. In the second part of the thesis, we study the slope of fibred surfaces. We give new proofs, using the generalised Cornalba-Harris Theorem, of some known results, and we prove a conjecture on the slope of double fibrations. Although the second part is the one that contains the "results" in the proper sense, we do not consider the first one just as a technical part, because the Cornalba-Harris Theorem, and its generalisation, have a considerable interest on their own, and can have applications in a wider range of topics. In particular it is worth to mention that this technique could be applied easily to the study of the numerical invariants of higher dimensional fibrations, such as for example fibred threefolds.

The Cornalba-Harris Theorem

The idea of the method is the following. Consider a family of polarised complex algebraic varieties. More precisely, let $f\colon X \longrightarrow T$ be a flat proper morphism of complex varieties (to fix ideas suppose that X and T are smooth), with a line bundle L on X whose restrictions to the *general* fibres of f give embeddings. Suppose that the Hilbert points of these embeddings are semi-stable in the sense of GIT. Then *the semi-stability assumption translates into the existence of a line bundle on the base T, together with a non-vanishing section of it.* This produces in particular an element in the effective cone of the base T. When T is a curve, the consequence is a non-trivial inequality holding between the degrees of certain naturally defined rational classes of divisors on it, and eventually an inequality involving the relative invariants of f.

In [CH88] this theorem is used to find a basic inequality holding in the rational Picard group of $\overline{M}_g$, the moduli space of stable curves of genus g. This implies a necessary and sufficient condition for a combination of the Hodge divisor λ and the boundary divisor δ to be nef outside the boundary $\partial\overline{M}_g$. In fact Cornalba and Harris can prove with their theorem only the nef-ness outside the hyperelliptic locus $\overline{H}_g$ in $\overline{M}_g$. Inside this locus, the inequality is obtained by directly computing the structure of the rational Picard group of $\overline{H}_g$.

The main idea of the generalisation of the Cornalba-Harris Theorem is to drop the assumption that the line bundle gives an *embedding* on the general fibres, and to consider arbitrary morphisms, or even rational maps. In order to do this, we need to introduce a suitable generalisation of Hilbert stability for a variety with a map in a projective space.

Assuming that this generalised semi-stability holds for the morphisms induced by the line bundle L on the general fibres, the argument of Theorem (1.1) of [CH88] works with almost no changes, and still gives as a consequence a naturally defined effective divisor on the base T. When T is a curve, and more specifically when f is a fibred surface, we can derive some explicit inequalities on the rational classes of divisors on it, which we exploit in the applications in the second chapter.

This generalisation sounds a little unnatural because, as GIT is mainly used to construct moduli spaces, GIT stability is usually defined for polarised varieties, i.e. for varieties with a line bundle whose associated morphism encodes all the informations about the variety, as in the case of the Hilbert points. However, in the Cornalba-Harris Theorem, as we mentioned above, the semi-stability is used to find relations between the classes of divisors on the base T, and it turns out that we can prove interesting statements in spite of (or rather thanks to) this weakening of the assumptions.

The semi-stability of morphisms to projective spaces is not easy to verify. In the case of curves, there is a sufficient condition which turns out to be very useful. It is the concept of *linear stability*, introduced for embeddings in [Mum77]. We give the definition for arbitrary maps in projective spaces, and we prove, following [ACGH], that for irreducible curves linear stability implies Hilbert stability. Thanks to this, we can prove Hilbert stability for projections of the canonical image of a non-hyperelliptic curve from a point. This is a key result for the computations on non-Albanese fibrations made in the second chapter. The stability of projections turns out to be closely related to the Clifford index of the curve.

The slope of fibred surfaces

Using the generalised Cornalba-Harris Theorem we can give a new proof of two results first discovered by Xiao: the slope inequality for arbitrary surfaces, and a bound for the slope of non-Albanese fibrations. Moreover, we prove via the Cornalba-Harris method a result of Barja (obtained again with Xiao's method) on the slope of double fibrations. It is worth to mention here that it is clear from our computations that the methods of Cornalba-Harris and Xiao, although they seem in principle quite different, present a remarkable analogy; indeed, they produce the same results starting from the same subsheaf of $f_*\omega_f$, although the arguments do not present a clear connection. It seems that the relation between these two methods should be better investigated and understood.

In the case of double cover fibrations, we can prove a sharp bound for the slope which was conjectured in 1998 by Barja himself; in this case, however, the main tool we use is the Algebraic Index Theorem.

We refer to section 2.1 for a detailed account of the problems and the known results about lower bounds for the slope of fibred surfaces. Here we recall that the sharp lower bound for arbitrary fibred surfaces (which we call *slope inequality*) has been proved in the eighties by Xiao and Cornalba-Harris. However, as mentioned above, the latter authors proved with their method only the case of non-hyperelliptic semi-stable fibred surfaces. They prove it applying their theorem to the relative canonical sheaf ω_f. Using the generalised version of the Theorem, together with a vanishing result, we are able to reprove in section 2.2 the slope inequality in its full generality. In particular our generalisation is essential in order to deal with the hyperelliptic case, where the restriction of ω_f to the general fibres does not induce an embedding, but the morphism obtained by composition of the hyperelliptic double cover of $\mathbb{P}^1$ with the Veronese embedding in $\mathbb{P}^{g-1}$.

A natural generalisation of the hyperelliptic case is the one of a fibred surface whose general fibres have an involution with quotient of arbitrary genus γ. We call this kind of fibrations *double* fibrations. Although this is not exactly true, a double fibration can be thought of as a double covering of a fibration of genus γ. Applying the Cornalba-Harris Theorem to this case, we can obtain a bound for the slope involving the invariants of the fibration (section 2.3.2).

In section 2.3 we prove a sharp bound for the slope of double fibrations with $g \geq 4\gamma + 1$, giving a positive answer to a conjecture of Barja (cf. [Bar]). Apart from some technicalities, the main ingredients of our argument are the Algebraic Index Theorem, and the slope inequality on the fibred surface of genus γ associated to f. We provide also a characterisation of the fibrations whose slope reaches the bound.

While, in the case of double fibrations, the general problem is to find how the properties of the general fibres influence the slope, in section 2.4 we address the problem of understanding the influence of a *global* invariant, the relative irregularity. We prove a result of Xiao using the Cornalba-Harris Theorem. For this computation the result on stability of projections proved in the first chapter is crucial. The use of the technique of Cornalba and Harris in this context seems to be quite promising. Indeed, we discuss further possible applications, and give some new evidence for a known conjecture on the influence of the relative irregularity on the slope.

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Notations and conventions

- We will work throughout over the complex field $\mathbb{C}$.
- A *variety* is an *integral separated scheme of finite type over* $\mathbb{C}$.
- Given a sheaf $\mathcal{F}$ over a variety T, we will designate by $\mathcal{F} \otimes \boldsymbol{k}(t)$ the geometric fibre of $\mathcal{F}$ over the point $t \in T$, while the symbol $\mathcal{F}_t$ indicates, as usual, the stalk of the sheaf $\mathcal{F}$ at t.
- In the second chapter we often make no distinction between divisors and line bundles on surfaces, and we use both the notations of tensor product and the additive one interchangeably. When we use intersection theory for line bundles it is understood that we are in fact using their first Chern classes.

Chapter 1

Stability of morphisms to $\mathbb{P}^s$ and applications

In this chapter we study two concepts of stability for a curve with a morphism in a projective space, Hilbert and linear stability; we prove that the first is implied by the second; eventually, we prove the Cornalba-Harris theorem.

The chapter is organised as follows. In the first section we define a generalisation of the classical Hilbert stability for polarised varieties. In section 1.2 we treat the concept of linear stability. For smooth non-hyperelliptic non-trigonal curves, we see in section 1.3 that a refinement of Clifford's theorem (Clifford plus) implies the linear stability for the projection of the canonical image from a point. Using the Clifford plus theorem we can also classify the curves with Clifford index up to 3. Section 1.4, contains the proof that for irreducible curves linear stability implies Hilbert stability. This result is proved in [ACGH]. This will allow us to exploit the results obtained in the preceding section for the application of the Cornalba-Harris Theorem in chapter 2. Section 1.5 is devoted to the proof of the generalisation of the Cornalba-Harris Theorem, together with some consequences.

1.1 Generalised Hilbert points and Hilbert stability

In this section we introduce the notion of "Hilbert point" of an arbitrary morphism from a variety to the projective space; and we give a notion of stability for these objects.

Results from Geometric Invariant Theory

Let G be a reductive algebraic group over $\mathbb{C}$ and V a finite dimensional complex representation of G. Recall the following definitions from Geometric Invariant Theory (GIT). See for reference [MFK94] and [Mum74].

Definition 1.1.1. *A nonzero element $x \in \mathbb{P}(V)$ is said to be*

- *GIT* semi-stable *if the closure of its orbit does not contain* 0*;*
- *GIT* stable *if its stabiliser is finite and its orbit closed;*
- *GIT* instable *if it is not semi-stable.*

Recall that a necessary and sufficient condition for the semi-stability of $x \in \mathbb{P}(V)$ is the existence of a G-invariant non-constant homogeneous polynomial $f \in \mathrm{Sym}(V^\vee)$ such that $f(v) \neq 0$, where $[v] = x$.

A very useful criterion for (semi-)stability is provided by the following result.

Lemma 1.1.2 (Hilbert-Mumford Criterion)**.** *Let G be a linearly reductive linear algebraic group, and let $\rho : G \to GL(V)$ be a finite-dimensional rational representation of G. A non-zero point $x \in \mathbb{P}(V)$ is semi-stable (resp. stable) if and only if for any one-parameter subgroup $\lambda : \mathbb{C}^* \to G$, $\rho\lambda(t)(x)$ does not converge to 0 (resp. diverges) for $t \to 0$.*

Generalised Hilbert points

Let X be a variety, with a non-degenerate rational map in a projective space $\psi : X \to \mathbb{P}^s$. So, if we set $L := \psi^*\mathcal{O}_{\mathbb{P}^s}(1)$, ψ corresponds to a linear subsystem $V \subseteq H^0(X, L)$. Notice that we do not require X to be smooth, nor irreducible. Moreover, the dimension of the image of ψ can be strictly smaller than the dimension of X. Fix $h \geq 1$ and call G_h the image of the homomorphism

$$H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(h)) = \mathrm{Sym}^h V \xrightarrow{\varphi_h} H^0(X, L^h).$$

Set $N = \dim G_h$ and take exterior powers

$$\wedge^N \mathrm{Sym}^h V \xrightarrow{\wedge^N \varphi_h} \wedge^N G_h = \det G_h.$$

If we identify $\det G_h$ with $\mathbb{C}$, the homomorphism $\wedge^N \varphi_h$ can be seen as a liner functional on $\wedge^N \mathrm{Sym}^h V$. Changing the isomorphism $\det G_h \cong \mathbb{C}$, it gets multiplied by a non-zero element of $\mathbb{C}$. Hence, we can see $\wedge^N \varphi_h$ as a well-defined element of $\mathbb{P}(\wedge^N \mathrm{Sym}^h V^\vee)$.

Definition 1.1.3. *We call $\wedge^N \varphi_h \in \mathbb{P}(\wedge^N Sym^h V^\vee)$, the* generalised h-th Hilbert point *associated to the couple (X, ψ).*

Remark 1.1.4. If ψ is an embedding, then for $h >> 0$ the homomorphism φ_h is surjective and it is the classical h-th Hilbert point associated to ψ. For large h, φ_h determines X as a subvariety of $\mathbb{P}^s$.

The standard action of $SL(s+1, \mathbb{C})$ on $\mathbb{C}^{s+1}$ induces a dual action on $H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$, and hence a linear action on $\wedge^N H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(h)) = \wedge^N \mathrm{Sym}^h H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$. So we have a natural induced action of $SL(s+1, \mathbb{C})$ on

$$\mathbb{P}(\wedge^N \mathrm{Sym}^h V^\vee).$$

Stability of generalised Hilbert points

Definition 1.1.5. *Let X be a variety, with a non-degenerate rational map in a projective space $\psi \colon X \longrightarrow \mathbb{P}^s$. We say that the h-th generalised Hilbert point of the couple (X, ψ) is semi-stable (resp. stable) if it is GIT semi-stable (resp. stable) with respect to the $SL(s+1, \mathbb{C})$-action described above.*

Consider the case when $\psi : X \to \mathbb{P}^s$ is a rational map which is not a morphism. We can extend it to a morphim $\widetilde{\psi}$ from the blow up $\widetilde{X}$ of X along the ideal of the base locus of ψ. It is immediate to check that (X, ψ) is generalised Hilbert (semi-)stable if and only if $(\widetilde{X}, \widetilde{\psi})$ is.

Remark 1.1.6. Let us suppose now that $\psi\colon X \to \mathbb{P}^s$ a *morphism.* Consider the factorisation of ψ through the (scheme-theoretic) image:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \mathbb{P}^s \\ \alpha \searrow & & \nearrow j \\ & \overline{X} & \end{array}$$

Set $L = \psi^*(\mathcal{O}_{\mathbb{P}^s}(1))$, $\overline{L} = j^*(\mathcal{O}_{\mathbb{P}^s}(1))$, and let $V \subseteq H^0(X, L)$ and $\overline{V} \subseteq H^0(\overline{X}, \overline{L})$ be the linear systems associated to ψ and j respectively. There is the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^h \overline{V} & = & \mathrm{Sym}^h V \\ \overline{\varphi}_h \downarrow & & \downarrow \varphi_h \\ H^0(\overline{X}, \overline{L}^h) & \xrightarrow{\eta} & H^0(X, L^h) \end{array}$$

where η is an inclusion, and the homomorphism $\overline{\varphi}_h$ is the h-th Hilbert point of the embedding $\overline{X} \hookrightarrow \mathbb{P}^s$. The generalised h-th Hilbert point of (X, ψ) is therefore naturally identified with the h-th Hilbert point of $(\overline{X}, j)$, as claimed.

Notice that this homomorphism is onto for large enough h. Indeed, it fits into the cohomology sequence

$$\dots \longrightarrow H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(h)) \xrightarrow{\overline{\varphi}_h} H^0(\overline{X}, \overline{L}^h) \longrightarrow H^1(\mathbb{P}^s, \mathcal{I}_{\overline{X}}(h)) \longrightarrow \dots$$

where $\mathcal{I}_{\overline{X}}$ is the ideal sheaf of $\overline{X}$. The last cohomology space vanishes for $h >> 0$ by Serre's vanishing theorem (cf. [Har77] Theorem III.5.2). So, in particular, $G_h = H^0(X, \overline{L}^h)$ for large enough h.

Hence, if X is reduced, the stability of the generalised Hilbert point of (X, ψ) corresponds to the stability of the classical Hilbert point of the immersion of the (set-theoretic) image $\overline{X} = \psi(X)$ in $\mathbb{P}^s$. We are therefore not introducing a really new concept.

However, the language of generalised Hilbert stability is very useful, and arises naturally, in the generalisation of the Cornalba-Harris Theorem which we present in section 1.5.

We now state an asymptotic version of this stability. This is the analogue of the (asymptotical) Hilbert stability for polarised varieties.

Definition 1.1.7. *We say that a rational map $\psi\colon X \longrightarrow \mathbb{P}^s$ is Hilbert stable (resp. semi-stable) if its generalised h-th Hilbert point is stable (resp. semi-stable) for infinitely many integers $h > 0$.*

1.2 Linear stability

In this section we introduce the notion of linear stability of a rational map from an irreducible variety to a projective space, which has been first defined for embeddings by Mumford in [Mum77], section 2.15. For irreducible curves embedded in a projective space, as we will see in section 1.4, it implies Hilbert stability; as it is in most cases easier to check than Hilbert stability itself, it will be very useful in the applications presented in chapter 2.

Let $\psi \colon X - \to \mathbb{P}^s$ be a rational map. Extend ψ by blowing up along the ideal of the base locus

$$\begin{array}{ccc} & \widetilde{X} & \\ \swarrow & & \searrow^{\widetilde{\psi}} \\ X & \overset{\psi}{--\to} & \mathbb{P}^s \end{array}$$

We define the image cycle $\psi_*(X)$ of ψ as the image cycle of $\widetilde{\psi}$, i. e. as $\widetilde{\psi}(\widetilde{X})$ times the degree of

$$\widetilde{\psi} \colon \widetilde{X} \to \widetilde{\psi}(\widetilde{X})$$

if $\widetilde{\psi}(\widetilde{X})$ has the same dimension as $\widetilde{X}$, 0 otherwise.

Definition 1.2.1. *Let X be a variety of dimension n, together with a non-degenerate rational map in a projective space $\psi : X \to \mathbb{P}^s$. The reduced degree of the pair (X, ψ) is*

$$\text{red.deg}(X, \psi) := \frac{\deg \psi_*(X)}{s + 1 - n},$$

where $\psi_*(X) \subset \mathbb{P}^s$ is the image cycle.

If $\mathcal{E}$ is the linear system inducing the morphism ψ, and V is the associated subspace of $H^0(X, \psi^*\mathcal{O}_{\mathbb{P}^s}(1))$, we will indicate the reduced degree of (X, ψ) also with the notation $\text{red.deg}(X, \mathcal{E})$ or $\text{red.deg}(X, V)$, or simply $\text{red.deg}(V)$.

Definition 1.2.2. *With the same notations as above, we say that $\psi : X \to \mathbb{P}^s$ is linearly semi-stable (resp. stable) if for any projection π such that the image cycle $\pi_*(X)$ of X has dimension n, then*

$$\text{red.deg}(X, \pi \circ \psi) \geq \text{red.deg}(X, \psi)$$

(resp. $\text{red.deg}(X, \pi \circ \psi) > \text{red.deg}(X, \psi)$*).*

Linear (semi-)stability is a geometric property of the morphism ψ. Indeed, we can rephrase the definition in terms of the linear subspace $V \subseteq H^0(X, \psi^*\mathcal{O}_{\mathbb{P}^s}(1))$ associated to ψ as follows. The map $\psi : X \to \mathbb{P}^s$ is linearly semi-stable if and only if for any subspace $W \subseteq V$, calling π_W the associated projection, $\pi_W : \mathbb{P}^s \setminus \text{Ann}(W) \to \mathbb{P}^{\dim W - 1}$, the following inequality holds:

$$\frac{\deg(\pi_W \circ \psi)_*(X)}{\dim W - n} \geq \text{red.deg}(X, \psi).$$

1.2.1 Linear stability for smooth curves

From now on, we shall consider the case of *smooth curves*. So consider a smooth curve C and a non-degenerate rational map $\psi : C \to \mathbb{P}^s$. By eliminating the base points we can extend ψ to a morphism $\overline{\psi}$. By what observed above, the generalised Hilbert stability and the linear stability of (C, ψ) correspond to the ones of $(C, \overline{\psi})$. Hence, we can suppose that ψ has no base points. The image is a possibly singular irreducible curve $\overline{C}$. Consider the factorisation of ψ as

$$C \xrightarrow{\eta} \overline{C} \xrightarrow{j} \mathbb{P}^s,$$

where η is a finite morphism of degree a and j is the embedding of the image $\overline{C}$.

In analogy with what happens for the stability of the generalised Hilbert points, we see below that *the linear stability of ψ is equivalent to the linear stability of j.* Indeed, set

$$L = \psi^* \mathcal{O}_{\mathbb{P}^s}(1), \qquad \overline{L} = j^* \mathcal{O}_{\mathbb{P}^s}(1),$$

Let $p : \mathbb{P}^s \setminus \Lambda \to \mathbb{P}^k$ be a projection from a linear subspace Λ. Note that, being ψ non-degenerate, the image of C has dimension 1. Let π be the map $p \circ \psi : C \to \mathbb{P}^k$ and $\overline{\pi}$ the map $p \circ j : \overline{C} \to \mathbb{P}^k$. Observing that $\deg L = a \deg \overline{L}$ and that $\deg \pi^* \mathcal{O}_{\mathbb{P}^k}(1) = a \deg \overline{\pi}^* \mathcal{O}_{\mathbb{P}^k}(1)$, we see that

$$\text{red.deg}(C, \psi) = \frac{\deg L}{s} = a \frac{\deg \overline{L}}{s} = a\, \text{red.deg}(\overline{C}, j),$$

and that

$$\text{red.deg}(C, p \circ \psi) = a\, \text{red.deg}(\overline{C}, p \circ j).$$

Therefore it is clear that (C, ψ) is linearly (semi-)stable if and only if $(\overline{C}, j)$ is.

Suppose that ψ is induced by a *complete* linear system $|L|$ free from base points. Then we can translate the notion of linear stability in terms of the degree of the invertible subsheaves of L.

Proposition 1.2.3. *Let $\psi : C \to \mathbb{P}^s$ be a morphism induced by a line bundle L of degree d such that $|L|$ is base-point free. Then (C, ψ) is linearly semi-stable if and only if for any proper invertible subsheaf M of L*

$$\frac{\deg M}{h^0(C, M) - 1} \geq \frac{d}{s}. \tag{1.1}$$

(C, ψ) is linearly stable if and only if for any proper invertible subsheaf M of L (1.1) holds with strict inequality.

Proof. Suppose first that (C, ψ) is semi-stable. Let M be a proper invertible subsheaf of L, and let $\varphi_{|M|}$ be the associated map. By the definition of semi-stability, we have that

$$\frac{\deg (\varphi_{|M|})_* (C)}{h^0(C, M) - 1} \geq \frac{d}{s}.$$

But $\deg (\varphi_{|M|})_* (C) \leq \deg M$ (equality holding if M is generated by global sections), so inequality (1.1) holds.

On the other hand, suppose that the inequality (1.1) holds for any invertible subsheaf M of L. Let W be a linear subsystem of V. Consider the subsheaf M_W of L generated by W. Its degree is equal to the degree of the image cycle $(\pi_W)_*C$, while $h^0(C, M_W) \geq \dim W$. So

$$\frac{\deg((\pi_W)_*(C)}{\dim W - 1} \geq \frac{\deg(M_W)}{h^0(C, M_W) - 1} \geq \frac{d}{s}.$$

Replacing the inequalities with strict inequalities we obtain the second part of the statement. □

Remark 1.2.4. If L is a line bundle not globally generated, the conditions of the above proposition still are sufficient conditions for the (semi-)stability, but of course they are not necessary. Indeed, if L' is the moving part of L, (C, ψ) is lineraly (semi-) stable if and only if inequality (1.1) holds for any proper invertible subsheaf M of L'.

1.3 Consequences of Clifford's and Castelnuovo's Theorems

We now recall two classical results from the theory of divisors on curves, namely Clifford's Theorem and Castelnuovo's bound, and state some interesting consequences on the linear stability of the maps induced by the canonical system and by its linear subsystems.

Theorem 1.3.1 (Clifford). *Let C be a smooth curve of genus $g \geq 2$ and D an effective divisor of degree $d \leq 2g - 1$ on C. Then*

$$d \geq 2(h^0(C, D) - 1).$$

Equality holds if and only if one of these conditions holds.

1. *$D \sim 0$;*
2. *D is a canonical divisor;*
3. *C is hyperelliptic and $|D|$ is a multiple of the g^1_2 on C.*

Theorem 1.3.2 (Castelnuovo). *Let C be a smooth curve that admits a birational map onto a non-degenerate curve of degree d in $\mathbb{P}^r$. Then its genus g satisfies the inequality*

$$g \leq \frac{m(m-1)}{2}(r-1) + m\epsilon, \tag{1.2}$$

where

$$m = \left[\frac{d-1}{r-1}\right]$$

and

$$d - 1 = m(r-1) + \epsilon.$$

The proof of these theorems can be found for instance in [ACGH85], and in [GH78], together with a detailed study of the curves reaching Castelnuovo's bound, which are called *(extremal) Castelnuovo curves.*

Remark 1.3.3. For a plane curve $C \subset \mathbb{P}^2$ of degree d, Castelnuovo's bound corresponds to the inequality deriving from Plücker's Formula for the genus

$$g \leq \frac{(d-1)(d-2)}{2}.$$

As is well-known, equality holds if and only if C is non-singular. Hence, plane extremal Castelnuovo curve are exactly the smooth ones.

For what concerns the curves in $\mathbb{P}^3$, we recall the following result, which will be useful later.

Lemma 1.3.4. *([ACGH85], Lemma III.2 pag.119) A Castelnuovo space curve C of degree $d = 2k$ is the complete intersection of a quadric and a surface of degree k. When $d = 2k+1$ then, for some line ℓ lying on the quadric, the sum $C + \ell$ is the complete intersection of a quadric and a surface of degree $k+1$.*

Linear stability of the canonical morphism

Immediate consequences of Clifford's Theorem are the following (see [ACGH] chap.14 sec.2):

Corollary 1.3.5. *If C is a non-hyperelliptic curve of genus $g \geq 3$, the canonical embedding of C in $\mathbb{P}^{g-1}$ is linearly stable. If C is hyperelliptic, the morphism induced by $|K_C|$ is linearly semi-stable, but not stable.*

Proof. Just observe that $\text{red.deg}(C, \varphi_{|K_C|}) = 2$; then apply Proposition 1.2.3 and Clifford's Theorem. □

Corollary 1.3.6. *If C is a smooth curve of genus $g \geq 1$ and L is a line bundle on C of degree $d > 2g$, the embedding induced by L is linearly stable.*

Proof. By Proposition 1.2.3 we can consider only invertible proper subsheaves M of L. If M is non-special, by the Riemann-Roch theorem $h^0(C, M) = \deg M - g + 1$, so

$$\frac{\deg M}{h^0(C,M)-1} = \frac{h^0(C,M)-1+g}{h^0(C,M)-1} > \frac{h^0(C,L)-1+g}{h^0(C,L)-1} = \frac{\deg L}{h^0(C,L)-1}.$$

If, on the other hand, M is special, then Clifford's Theorem implies that

$$\frac{\deg M}{h^0(C,M)-1} \geq 2,$$

while

$$\frac{\deg L}{h^0(C,L)-1} = \frac{\deg L}{\deg L - g} = 1 + \frac{g}{\deg L} \leq 1 + \frac{g}{g+1} < 2.$$

□

1.3.1 Refinements of Clifford's Theorem

We now present a refinement of Clifford's Theorem, proved by Beauville in [Bea82] and Reid in [Rei88]. It is a consequence of the general position theorem (see [ACGH85] III.1 and [Mir95] VII.3) and of Clifford's Theorem.

Theorem 1.3.7 (Clifford plus)**.** *Let C be a smooth curve and D a special divisor of degree d and $h^0(C, D) = r + 1$. Then, one of the following conditions holds.*

1. *The map $\varphi_{|D|}$ factors through a double cover $\pi : C \to \Gamma$ over a smooth curve Γ of genus γ. In this case either*
$$d \geq 2r + 2\gamma, \tag{1.3}$$
or
$$d \geq 4r. \tag{1.4}$$

2. *The map $\varphi_{|D|}$ factors through a cover of degree ≥ 3. In this case $d \geq 3r$.*

3. *The map $\varphi_{|D|}$ is birational onto its image and*
   - *(a) if $2D$ is special, then $d \geq 3r - 1$;*
   - *(b) if $2D$ is non-special, then $2d \geq 3r - 1 + g$.*

Proof. Consider the factorisation of the map $\varphi_{|D|}$ through the normalisation $\widetilde{C}$ of the image of C

$$\begin{array}{ccc} C & \xrightarrow{\varphi_{|D|}} & \mathbb{P}^r \\ \alpha \searrow & & \nearrow j \\ & \widetilde{C} & \end{array}$$

where α is a finite morphism of degree a, and j is birational. Let Δ be the line bundle $j^*\mathcal{O}_{\mathbb{P}^r}(1)$ on $\widetilde{C}$.

If $a = 2$ we are in situation (1); let us distinguish two cases
a) If Δ is special, then by Clifford's Theorem applied to it

$$\deg \Delta \geq 2(h^0(\widetilde{C}, \Delta) - 1) = 2r;$$

by assumption $d = 2 \deg \Delta$, so we get $d \geq 4r$.
b) If Δ is non-special, then by the Riemann-Roch Theorem

$$d = 2 \deg \Delta = 2(h^0(\widetilde{C}, \Delta) - 1 + \gamma) \geq 2r + 2\gamma.$$

If $a \geq 3$, we are in case (2). As j is non-degenerate, $\deg \Delta \geq r$, and by assumption $d \geq 3 \deg \Delta$, so $d \geq 3r$ as claimed. Notice that in fact, arguing as in the double cover case, above, we can find that, if γ is the genus of the quotient curve, either $d \geq ar + a\gamma$, or $d \geq 2ar$.

If $a = 1$, we are in case (3). Let us consider the homomorphism

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) = \mathrm{Sym}^2 H^0(\widetilde{C}, \widetilde{D}) \longrightarrow H^0(\widetilde{C}, 2\widetilde{D}). \tag{1.5}$$

where $\widetilde{D} = j^*\mathcal{O}_{\mathbb{P}^r}(1)$. From the general position theorem we know that d points in general position on $\widetilde{C}$ impose at least $\min\{d, 2r-1\}$ conditions on quadrics, i. e. that the rank of this homomorphism is at least

$$r + 1 + \min\{d, 2r-1\} = \min\{r+d, 3r\}.$$

Therefore, $h^0(C, 2D) \geq h^0(\widetilde{C}, \widetilde{D}) \geq \min\{r+d, 3r\}$. As D is special, this last number is $3r$. If $2D$ is special, by Clifford's Theorem

$$2d \geq 2(h^0(C, 2D) - 1) \geq 6r - 2.$$

On the other hand, if $2D$ is not special, by the Riemann-Roch Theorem

$$2d - g + 1 = h^0(C, 2D) \geq 3r.$$

□

1.3.2 Projection of the canonical image from a point

Thanks to the Clifford plus Theorem, we are able to prove linear stability of projections of the canonical image from a point, i.e. linear stability of morphisms induced by linear subsystems of $|K_C|$ of projective dimension $g-2$. This will have applications in section 2.4. As we will see in next section, this result implies a known result about curves with Clifford index equal to 1.

Theorem 1.3.8. *Let C be a smooth non-hyperelliptic non-trigonal curve of genus $g \geq 5$. Embed C in $\mathbb{P}^{g-1}$ via its canonical system. The projection from a point outside the canonical image of C is a linearly stable morphism unless $g = 6$ and C is a smooth plane curve. In this case the projection from a point is linearly semi-stable.*

Proof. Call $V \subset H^0(K_C)$ the (non-complete) linear system associated to the projection. Let $W \subset V$ be a linear subsystem of projective dimension ≥ 1. Let L_W be the line bundle generated by W. If $\deg L_W = 2g-2$ then

$$\text{red.deg}(W) = \frac{2g-2}{\dim W - 1} > \frac{2g-2}{g-2} = \text{red.deg}(V).$$

Otherwise, recalling that C in non-hyperelliptic, Clifford's Theorem implies that

$$\deg L_W \geq 2h^0(L_W) - 1 \geq 2\dim W - 1.$$

If one of the two equalities above is strict, then

$$\text{red.deg}(W) = \frac{\deg L_W}{\dim W - 1} \geq 2 + \frac{2}{\dim W - 1} > \text{red.deg}(V).$$

Suppose on the contrary that $\deg L_W = 2h^0(L_W) - 1 = 2\dim W - 1$ (so in particular W is a complete linear system). Then

$$\text{red.deg}(W) = \frac{\deg L_W}{h^0(L_W) - 1} = 2 + \frac{1}{h^0(L_W) - 1}$$

and this last quantity is strictly bigger than red.deg(V) if and only if $2h^0(L_W) < g$, i.e. $\deg L_W < g-1$. Hence, we can suppose $\deg L_W \geq g-1$.

We now apply the Clifford plus Theorem to L_W. If the associated map is not birational, we are in case (1) or (2) of Theorem 1.3.7, and we have one of these inequalities (recall that C is non-hyperelliptic)

$$\text{red.deg}(W) \geq 2 + \frac{2}{\dim W - 1} > \text{red.deg}(V),$$

$$\text{red.deg}(W) \geq 4 > \text{red.deg}(V) \quad \text{or} \quad \text{red.deg}(W) \geq 3 > \text{red.deg}(V).$$

Suppose that the morphism is birational. If $|L_W^2|$ is non-special, then by Theorem 1.3.7

$$2\deg L_W \geq 3(h^0(L_W)-1) + g - 1,$$

and we obtain

$$4h^0(L_W) - 2 \geq 3h^0(L_W) + g - 4 \quad \Longrightarrow \quad h^0(L_W) \geq g-2.$$

Recalling that we have assumed $\dim W = h^0(L_W)$, and $\deg L_W = 2h^0(L_W) - 1$, the only possibility is that $h^0(L_W) = g-2$ and $\deg L_W = 2g-5$. Serre's duality and the Riemann-Roch Theorem show that $h^0(L_W) - h^0(K_C L_W^{-1}) = g-4$. It follows that $|K_C L_W^{-1}|$ is a g_3^1 on C.

The line bundle L_W^2 is special if and only if it is isomorphic to the canonical bundle; so in this case L_W should be a theta-characteristic with $2h^0(L_W) = g$. Applying again the Clifford plus Theorem, we obtain

$$g - 1 = \deg L_W \geq 3h^0(L_W) - 4 = \frac{3}{2}g - 4 \;\Rightarrow\; g \leq 6.$$

Since g is even, it remains to deal with the case $g = 6$. Then $h^0(L_W) = 3$, and the Plücker formula assures that the map associated to L_W is an embedding in $\mathbb{P}^2$. Notice moreover that in this last case C, as any smooth plane curve, is an extremal Castelnuovo curve. □

Remark 1.3.9. From the proof of the above proposition we can derive that if $g \geq 5$ the projection of a canonical non-trigonal curve $C \subset \mathbb{P}^{g-1}$ from a point P *contained in it* is linearly stable. Indeed,

$$\text{red.deg}(K_C(-P)) = \frac{2g-3}{g-2} = 2 + \frac{1}{g-2},$$

which is strictly smaller than the reduced degree of a projection from a point outside the curve; so we can apply the same argument as in the proposition. Moreover, in the case of a plane quintic strict inequality still holds, because

$$\text{red.deg}(W) = \frac{5}{2} > \frac{2g-3}{g-2} = \frac{9}{4}.$$

Projection for trigonal curves

If C is a *trigonal* curve, the projection from a point outside the canonical image can be linearly unstable. Indeed, consider any effective divisor $P_1 + P_2 + P_3$ belonging to the g^1_3 on C (the P_i's need not be distinct). By the geometric Riemann-Roch Theorem these points span a line $\ell \subset \mathbb{P}^{g-1}$. Let P be a point of ℓ disjoint from C. The projection π from P is a birational morphism, as can be checked directly, or using Proposition 1.3.17 below. The image of π, $\overline{C}$ has a triple point R. If we consider the projection ψ from R, we have

$$\text{red.deg}(\overline{C}, \psi) = \frac{2g-5}{g-3} < \frac{2g-2}{g-2} = \text{red.deg}(C, \pi),$$

for $g \geq 5$. The map ψ corresponds to the linear subsystem $H^0(K_C(-P_1 - P_2 - P_3))$ of the canonical system. From the proof of the above theorem, we see that this is the only problem for trigonal curves. More precisely, consider the projection from a point outside the canonical image of a trigonal curve C and call V the associated linear system. If W is a linear subsystem of V such that

$$\text{red.deg}(W) \leq \text{red.deg}(V)$$

then either $W = V$ or $W = H^0(K_C(-P_1 - P_2 - P_3))$, for $P_1 + P_2 + P_3$ belonging to the g^1_3. In this last case P clearly belongs to the line spanned by the P_i's. Hence, we have proven the following

Proposition 1.3.10. *If $C \subset \mathbb{P}^{g-1}$ is a trigonal canonical curve of genus $g \geq 5$, the projection from a point not contained in a 3-secant line is linearly stable.*

Note that the variety of trisecant lines of $C \subset \mathbb{P}^{g-1}$ has dimension 2, because a trigonal curve of genus greater or equal to 5 possesses a unique g^1_3, hence a general point of $\mathbb{P}^{g-1}$ satisfies the condition of the proposition.

1.3.3 Relation with the Clifford index

Given a line bundle L over a smooth curve C, we define its Clifford index $\text{Cliff}(L)$ as $\text{Cliff}(L) = \deg L - 2(h^0(L) - 1)$. We now define the Clifford index of a curve, first introduced by H. H. Martens in [Mar67].

Definition 1.3.11. *The* Clifford index *of a curve C of genus $g \geq 4$ is the integer:*

$$\text{Cliff}(C) = \min\{\text{Cliff}(L) \ \mid \ L \in \text{Pic}(C),\, h^0(L) \geq 2,\, h^1(L) \geq 2\}.$$

When $g = 2$ we set $\text{Cliff}(C) = 0$*; when $g = 3$ we set* $\text{Cliff}(C) = 0$ *or* 1 *according to whether C is hyperelliptic or not.*

If a divisor in the above conditions achieves the minimum, we say that it *computes* the Clifford index, while a divisor with h^0 and h^1 greater or equal to 2 is said to *contribute* to the Clifford index.

Remark 1.3.12. Notice that if D is a divisor that computes the Clifford index, by the Riemann-Roch Theorem the residual divisor $K_C - D$ also achieves the minimum. It is therefore equivalent to consider the minimum of $\deg L - 2(h^0(L)-1)$ running over all divisors $L \in \mathrm{Pic}(C)$ with $h^0(L) \geq 2$ and $\deg L \leq g-1$. Moreover, observe that given any (not necessarily complete) linear series g^r_d on C, and any divisor $D \in g^r_d$, $d-2r \geq d-2(h^0(D)-1)$. Therefore

$$\mathrm{Cliff}(C) = \min\{d - 2r \,|\, \text{ there exist a } g^r_d \text{ on } C, \text{ such that } r \geq 1,\ d \leq g-1\}.$$

Clifford's Theorem says that the curves with Clifford index 0 are exactly the hyperelliptic ones. Moreover, Brill-Noether theory ([ACGH85], Chapter V) shows that $\mathrm{Cliff}(C) \leq [(g-1)/2]$, and that equality holds if C is general in moduli.

Recall that the *gonality* $\mathrm{gon}(C)$ of a curve C is defined to be the minimum degree of surjective morphisms from C to $\mathbb{P}^1$, i.e. the minimum of degrees of rational functions on C.

The Clifford index of a curve is related to its gonality in the following way. For a curve C general in moduli $\mathrm{gon}(C) = \mathrm{Cliff}(C) + 2$ (as proved in [Bal86]), while a special curve can have gonality equal to $\mathrm{Cliff}(C) + 3$ (see [CM91] and [Mar68]). Hence,

$$\mathrm{gon}(C) - 3 \leq \mathrm{Cliff}(C) \leq \mathrm{gon}(C) - 2.$$

Gonality and Clifford index of plane curves

In the case of *smooth* plane curves, the following result holds.

Lemma 1.3.13. *Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 2$. Then*

1. *The gonality of C is $d-1$, and every g^1_{d-1} on C is cut out by the pencil of lines through some fixed point of C.*
2. $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3 = d - 4$.

Proof. The first part of the lemma is a classical result, known to M. Noether. For modern proofs see for instance [Har02] and [Nam79]. The second part follows immediately from the first, because the Clifford index of the g^1_{d-1} is $d-3$, while the Clifford index of the linear system that gives the embedding of C in $\mathbb{P}^2$ is $d-4$. □

There are some generalisations of these results to irreducible plane curves with nodes and cusps due to Coppens and Kato (cf. [CK90] and [Cop91]).

Curves of Clifford index 1

Trigonal curves and smooth plane quintics, which are, together with hyperelliptic curves, the ones we have excluded in Proposition 1.3.8, have Clifford index exactly 1. In fact, Proposition 1.3.8 implies the following statement (cf. [Mar68], 2.51):

Proposition 1.3.14. *The curves with Clifford index 1 are the trigonal ones and the smooth plane quintics.*

We present here an alternative proof of this statement, which relies on a result of H. H. Martens (cf. [ACGH85] Theorem IV 5.1, and [Mar67]) and on a refinement of it due to Mumford (Theorem IV 5.2 of [ACGH85]).

Proof. Let us suppose that $\mathrm{Cliff}(C) = 1$. Then C possesses a complete g^r_d with $d - 2r = 1$. Let suppose that d and r are minimal with respect to this property. Let $W^r_d(C)$ be the subvariety of $\mathrm{Pic}(C)$ parametrising complete linear series of degree d and dimension at least r. Martens' theorem assures that

$$\dim W^r_d(C) \leq d - 2r - 1 = 0,$$

As $W^r_d(C)$ is non empty by assumption, we conclude that $\dim W^r_d(C) = 0$. Mumford's Theorem says that if $\dim W^r_d(C) = d - 2r - 1$ then one of the following holds

- $r = 1$ and $d = 3$, so C is trigonal.
- $r = 2$ and $d = 5$, so C is a smooth plane quintic.
- $r = 2$ and $d = 6$. In this case C is bi-elliptic (i.e. a double cover of an elliptic curve), and the family of g^2_3's is obtained composing the bi-elliptic involution with an embedding of the elliptic curve in $\mathbb{P}^2$ via a divisor of degree 3.

Clearly the first two are the only cases in which $d - 2r = 1$. □

Remark 1.3.15. As the Clifford index of a curve C measures how large is the ratio between the degree and the dimension of special linear series on C, it seems natural to guess that the curves with higher Clifford index have linearly stable projections from positive-dimensional subspaces of $\mathbb{P}^{g-1}$. However, this guess is false. The problem is that the Clifford index does not control the divisors having H^1 of dimension 1. Indeed, consider a non-hyperelliptic curve C with *arbitrary* Clifford index, and let $D = P_0 + \ldots P_k$ be an effective divisor consisting of $k + 1$ points that impose independent conditions on $H^0(K_C)$. Consider a section φ of $H^0(K_C)$ not vanishing at anyone of the P_i's (a general section will do). The linear subsystem V of $H^0(K_C)$ spanned by $H^0(K_C(-D))$ and by φ has no base points by construction, and has dimension $g - k$. Hence, V induces the projection of the canonical image of C from a subspace of projective dimension $k - 1$ disjoint from it. As soon as $k > 1$ this projection is linearly unstable, because

$$\mathrm{red.deg}(V) = \frac{2g-2}{g-k-1} > \mathrm{red.deg}(K_C(-D)) = \frac{2g-3-k}{g-k-2}.$$

Note that $h^1(K_C(-D)) = 1$, and hence $K_C(-D)$ is one of the divisors that do not contribute to the Clifford index of C.

From this remark we see that, if we want to prove the linear stability of the projection from a subspace of positive dimension, we need to have some condition on the subspace.

Curves of Clifford index 2 and 3

Using the Clifford plus Theorem, we can characterise the curves with Clifford index 2 and 3. In [Mar80] (Beispiel 8 and Beispiel 9), G. Martens obtains the same result, with a different argument.

Proposition 1.3.16. *Let C be a smooth curve of genus $g \geq 3$.*

1. *If* $\mathrm{Cliff}(C) = 2$*, then C is of one of the following kinds*
   (a) *a 4-gonal curve;*
   (b) *a plane sextic ($g = 10$).*
2. *If* $\mathrm{Cliff}(C) = 3$*, then C is of one of the following kinds*
   (a) *a 5-gonal curve;*
   (b) *a smooth plane curve of degree 7 ($g = 15$);*
   (c) *a complete intersection of two cubic surfaces in $\mathbb{P}^3$ ($g = 10$).*

Moreover, every 4-gonal curve, with the exception of smooth plane quintics, is of Clifford index 2, and every 5-gonal curve, with the exception of smooth plane sextics, is of Clifford index 3.

Proof. 1) Suppose that $\mathrm{Cliff}(C) = 2$, so there exist a special divisor D inducing a (complete) g^r_d on C with $d = 2r + 2$. We apply the Clifford plus Theorem to D. As observed in Remark 1.3.12, we can suppose that $d \leq g - 1$, because the other cases are covered by adjunction.

Suppose that $\varphi_{|D|}$ factors through a double cover over a genus γ curve. If inequality (1.3) holds, γ has to be 1; hence C is a bi-elliptic curve, and has gonality 4. If on the other hand inequality (1.4) holds, $r = 1$ and C is hyperelliptic.

If $\varphi_{|D|}$ factors through a cover of degree $a \geq 3$ a has to divide $2r + 2$ and it has to be $2r + 2 \geq ar$. Hence the the only possibilities are

- $a = 4$ and $r = 1$, so C is 4-gonal;
- $a = 3$, $r = 2$. In this case C is trigonal because it is a triple cover of a conic in $\mathbb{P}^2$.

Let us suppose that $\varphi_{|D|}$ is a birational map (hence $r \geq 2$).
Then if $2D$ is special

$$2r + 2 = d \geq 3r - 1 \implies r \leq 3.$$

- If $r = 3$, then C is a space curve of degree 8. Using Castelnuovo's bound, we get $g \leq 6$. On the other hand, the assumption that $2D$ be special implies that $d < g - 1$, so $g > 9$. Hence this case is impossible.
- If $r = 2$, then C is a plane sextic. The Plucker formula gives $g \leq 10$, and as observed above it has to be $g > 9$, so $g = 10$ and C is a smooth plane sextic. Such a curve has indeed Clifford index 2, by Lemma 1.3.13.

If, on the other hand, $2D$ is non-special, we have necessarily that $d = g-1$ and $2D \sim K_C$ (hence $r = (g-3)/2$). From Clifford plus Theorem we obtain the inequality

$$2r + 2 = d \geq \frac{3}{2}r - \frac{1}{2} + \frac{g}{2} \implies r \geq g - 5.$$

In conclusion, we get $g \leq 7$. The possible cases are:

- $g = 7$, $d = 2$. In this case C has a plane singular model of degree 6. Projecting from a singular point of $\overline{C}$, we obtain a g^1_α with $\alpha \leq 4$, so C has gonality at most 4;

- $g = 5$, $r = 1$. In this case $d = 4$ and C is 4-gonal.

2) In the case $\mathrm{Cliff}(C) = 3$, there exist a divisor D on C with $d = 2r + 3$. We argue exactly as in the first part. So in particular we suppose again that $d \leq g - 1$.

In this case it is impossible that $\varphi_{|D|}$ factors through a double cover; indeed, by inequality (1.3), C should be hyperelliptic or bi-elliptic (hence of Clifford index smaller than 3).
If $\varphi_{|D|}$ factors through a cover of degree $a \geq 3$, using the conditions $a \mid 2r+3$ and $2r+3 \geq ar$, we see that the only possibilities are

- $a = 3$, $r = 3$. In this case C is a triple cover of a degree 3 space curve, which is necessarily rational, therefore C is trigonal (hence of Clifford index 1).

- $a = 5$, $r = 1$, hence C is 5-gonal.

If $\varphi_{|D|}$ is birational, we distinguish the two cases of Theorem 1.3.7.
If $2D$ is special $2r + 3 = d \geq 3r - 1$, so $r \leq 4$. Notice moreover that it has to be $d < g - 1$. We analyse the possible cases

- $r = 4$, $d = 11$. In this case Castelnuovo's bound gives $g \leq 12$, so we get $d < g - 1 \leq 11$, a contradiction.

- $r = 3$, $d = 9$. We will show that this case is also impossible. Castelnuovo's bound implies $g \leq 12$, while on the other hand $g \geq 11$. By [Har77] IV Ex. 6.4 we see that there don't exist space curves of degree 9 and genus 11. If $g = 12$ then C is an extremal Castelnuovo curve, so by Lemma 1.3.4 it lies on a quadric surface Q, and on a quintic surface. One of the two rulings of Q is a family of 4-secants of C, so C has gonality ≤ 4 and in particular $\mathrm{Cliff}(C) \leq 2$.

- $r = 2$, $d = 7$. In this case C has a plane model of degree 7. Plücker's formula gives $g \leq 15$, and on the other hand it has to be $g \geq d + 2 = 9$. If $g = 15$ (and hence the planar model of C is smooth), then by Lemma 1.3.13 $\mathrm{Cliff}(C) = 3$. If on the other hand $g < 15$, projecting from one of the singular points of the plane model of C, we obtain a g^1_α with $\alpha \leq d - 2 = 5$, hence $\mathrm{gon}(C) \leq 5$, and we are led back to cases we have already treated[1].

[1]Note that a Theorem of Coppens and Kato ([CK90]) assures than in most cases, provided that the singularities are nodes or ordinary cusps, the gonality of C is exactly 5.

If $2D$ is non-special then the inequality of Clifford plus Theorem is

$$2r + 3 \geq \frac{3}{2}r - \frac{1}{2} + \frac{g}{2} \implies r \geq g - 7.$$

Moreover, it has to be $d = g - 1$, so

$$\frac{g-4}{2} = r \geq g - 7 \implies g \leq 10.$$

- If $g = 10$, $r = 3$ and $d = 9$. Then there are two possibilities for C, as shown in [Har77], IV 6.4.3. Either C is the complete intersection of two cubic surfaces in $\mathbb{P}^3$, or it lies on a quadric surface with a ruling consisting of trisecants of C. So in this last case C is trigonal, hence $\text{Cliff}(C) = 1$.
- If $g = 8$, $r = 2$, then C has a singular plane model of degree 7, so its gonality is smaller or equal than 5 as observed above.

The last part of the proposition follows immediately from the first two points and from the classification of curves with Clifford index 1. □

1.3.4 Further applications

In the application to non-Albanese fibrations treated in section 2.4, we shall need to know whether the projection from a linear subspace of $\mathbb{P}^s$ is a birational morphism or not. We therefore state the following result, which is itself a consequence of Castelnuovo's bound (see [ACGH85], Exercise B-7).

Proposition 1.3.17. *If a smooth curve C of genus g has a base point free linear system $\mathcal{E}$ of dimension r and degree $2r + c$, with $0 \leq c \leq r - 2$, then either:*

1. *the map induced by $\mathcal{E}$ is not birational and factors through a double cover over a curve of genus at most $c/2$, or*
2. *the map induced by $\mathcal{E}$ is birational onto its image and either:*
   - *(a) $g \leq r + 2c$, or*
   - *(b) $g = r + 2c + 1$ and C is trigonal if $c > 0$, while $\mathcal{E} = |K_C|$ if $c = 0$.*

Proof. Suppose that the map induced by $\mathcal{E}$ is not birational. Consider its factorisation through the normalisation of the image

$$C \xrightarrow{\alpha} \widetilde{C} \xrightarrow{\eta} \mathbb{P}^r;$$

where α is a finite morphism of degree $a > 1$, and η is a birational morphism. Therefore $\widetilde{C}$ is a smooth curve with a linear system $\widetilde{\mathcal{E}}$ of degree $(2r + c)/a$ and dimension r. As η is non-degenerate, it has to be $(2r + c)/a \geq r$, so a has to be 2. Applying Castelnuovo's bound to $\widetilde{C}$ and $\widetilde{\mathcal{E}}$, we see that $m = 1$ and $\epsilon = c/2$, so we get $g(\widetilde{C}) \leq c/2$.

Suppose on the other hand that the map induced by $\mathcal{E}$ is birational. Then we can apply Castelnuovo's bound directly to C and $\mathcal{E}$. In this case

$$m = \left[\frac{2r+c-1}{r-1}\right] = 2 + \left[\frac{c+1}{r-1}\right] = \begin{cases} 3 & \text{if } c = r-2 \\ 2 & \text{otherwise} \end{cases}$$

If $c < r-2$, then $m = 2$, $\epsilon = c+1$, and inequality (1.2) becomes

$$g \leq r + 2c + 1.$$

If $c = r-2$ then $m = 3$, $\epsilon = 0$ and we obtain again

$$g \leq 3r - 3 = r + 2c + 1.$$

If equality holds, C is an extremal Castelnuovo curve, so it is clear that if $c = 0$ it has to be a canonical curve, while if $c > 0$ statement (b) follows from Corollary III.2.6 of [ACGH85]. □

1.4 Linear stability implies Hilbert stability

In this section we shall prove that the linear stability of an embedding of an irreducible curve C in a projective space implies the stability of its h-th Hilbert point for infinitely many positive integers h. This is a result contained in [ACGH] (Theorem (2.2)). We report it here due to its importance for our applications, and to the lack of references. The proof uses Gieseker's techniques ([Gie82]).

First we adapt the Hilbert-Mumford criterion (Lemma 1.1.2) to one which suits better our situation, reducing the proof of stability to proving a geometric property of the linear system associated to the morphism.

Let us first settle some terminology. Pick a basis $\{X_0, \ldots, X_s\}$ of the linear system V. Let $\rho_0, \ldots, \rho_s$ be integers not all 0 such that $\sum_i \rho_i = 0$. The weight of a monomial $\prod X_i^{m_i}$ (with respect to $\{X_0, \ldots, X_s\}$ and $\rho_0, \ldots, \rho_s$) is defined to be $\sum m_i \rho_i$, while the weight of an element of $\mathrm{Sym}^h V$ is the maximum of the weights of the monomials that compose it. The weight of an element of G_h is the minimum of the weights of the elements of $\mathrm{Sym}^h V$ that are mapped to it by ψ_h. Finally, the weight of a basis of G_h is the sum of the weights of its elements.

Lemma 1.4.1. *The h-th generalised Hilbert point of $\psi : C \to \mathbb{P}^s$ is semi-stable (resp. stable) if and only if, for any choice of the basis $\{X_0, \ldots, X_s\}$ and of integers $\rho_0, \ldots, \rho_s$ as above, G_h has a basis of non-positive (resp. negative) weight.*

For the proof see [ACGH], chap 14 Lemma (1.2) or [Gie77], Theorem 1.1 and the subsequent discussion.

We now come to the proof of the main result of this section.

Theorem 1.4.2. *Let C be an irreducible curve embedded in $\mathbb{P}^s$ by a very ample linear system V. Suppose that $(C, \varphi_{|V|})$ is linearly stable. Then it is Hilbert stable.*

Proof. We shall use the criterion provided by Lemma 1.4.1. Thus, given a basis $\{X_0, \dots, X_s\}$ of V and integers $\rho_0 \le \dots \le \rho_s$, not all zero, such that $\sum \rho_i = 0$, we must find a basis of negative weight for G_h for infinitely many h. Let L be the line bundle generated by V. We denote by V_i the subspace of V generated by $\{X_0, \dots, X_i\}$, and by d_i its degree. Let p and N be positive integers, to be chosen later, and let

$$0 = h_0 < \dots < h_l = s$$

be a finite sequence of integers, also to be chosen later. Denote by $W_{j,k}$ the image of the homomorphism

$$\mathrm{Sym}^{N(p-k)} V_{h_j} \otimes \mathrm{Sym}^{Nk} V_{j+1} \otimes \mathrm{Sym}^N V \longrightarrow H^0(C, L^{N(p+1)}) .$$

These subspaces clearly provide a filtration

$$W_{0,0} \subset W_{0,1} \subset \dots \subset W_{0,p-1} \subset W_{1,0} \subset \dots \subset W_{l-1,p-1} \subset W_{l,0} \subset H^0(C, L^{N(p+1)}) ,$$

and the weight of an element of $W_{j,k}$ does not exceed

$$q_{j,k} = N(p-k)\rho_{h_j} + Nk\rho_{h_{j+1}} + N\rho_s .$$

For large N, $W_{l,0} = G_{N(p+1)}$ has a basis of weight not exceeding

$$q_{0,0}w_{0,0} + q_{0,1}(w_{0,1} - w_{0,0}) + \dots + q_{l,0}(w_{l,0} - w_{l-1,p-1}) =$$

$$= w_{0,0}(q_{0,0} - q_{0,1}) + \dots + w_{l-1,p-1}(q_{l-1,p-1} - q_{l,0}) + N(p+1)\rho_s \dim G_{N(p+1)},$$

where $w_{j,k}$ denotes the dimension of $W_{j,k}$. All the terms in the last summand are negative except for the last one. Therefore, to estimate the weight of a basis of $G_{N(p+1)}$ from above, we need a good lower estimate for the $w_{j,k}$. The bound we need is provided by the result below. As usual, if U is a nonzero subspace of V, L_U stands for the subsheaf of L generated by U.

Lemma 1.4.3. *Let C be an irreducible curve of arithmetic genus $p_a \ge 1$, V a very ample linear system on C of degree d, and p a positive integer. Then there is an integer N_0, such that for any $N > N_0$, any integer k such that $0 \le k \le p$, and any non-zero subspaces U and Λ of V, the dimension of the image of*

$$\mathrm{Sym}^{N(p-k)} U \otimes \mathrm{Sym}^{Nk} \Lambda \otimes \mathrm{Sym}^N V \longrightarrow H^0(C, L^{N(p+1)})$$

is at least

$$N(p-k)\deg(L_U) + Nk\deg(L_\Lambda) .$$

Proof. To prove the lemma it suffices to show that

$$\mathrm{Sym}^{N(p-k)} U \otimes \mathrm{Sym}^{Nk} \Lambda \otimes \mathrm{Sym}^N V \longrightarrow H^0(C, L_U^{N(p-k)} \otimes L_\Lambda^{Nk} \otimes L^N) \qquad (1.6)$$

is onto for large N, since by the Riemann-Roch Theorem $H^0(C, L_U^{N(p-k)} \otimes L_\Lambda^{Nk} \otimes L^N)$ is a subspace of $H^0(C, L^{N(p+1)})$ of dimension

$$N(p-k)\deg(L_U) + Nk\deg(L_\Lambda) + Nd + 1 - p_a \ge N(p-k)\deg(L_U) + Nk\deg(L_V)$$

for Nd large enough. Denote by Z the image of

$$\mathrm{Sym}^{p-k}U \otimes \mathrm{Sym}^k\Lambda \otimes V \longrightarrow H^0(C, L_U^{p-k} \otimes L_\Lambda^k \otimes L)\,.$$

Since the images of $\mathrm{Sym}^{p-k}U$ and $\mathrm{Sym}^k\Lambda$ in $H^0(C, L_U^{p-k})$ and $H^0(C, L_\Lambda^k)$ generate by definition L_U^{p-k} and L_Λ^k, respectively, and V is very ample, the linear system $|Z|$ is a base-point free very ample linear subsystem of $|L_U^{p-k} \otimes L_\Lambda^k \otimes L|$. Thus the homomorphism

$$\mathrm{Sym}^N Z \longrightarrow H^0(C, L_U^{N(p-k)} \otimes L_\Lambda^{Nk} \otimes L^N) \tag{1.7}$$

is onto for large N. Since (1.6) is the composition of (1.7) and of the surjective homomorphisms

$$\mathrm{Sym}^{N(p-k)}U \otimes \mathrm{Sym}^{Nk}\Lambda \otimes \mathrm{Sym}^N H^0(C,L) \to \mathrm{Sym}^N(\mathrm{Sym}^{p-k}U \otimes \mathrm{Sym}^k\Lambda \otimes H^0(C,L))\,,$$
$$\mathrm{Sym}^N(\mathrm{Sym}^{p-k}U \otimes \mathrm{Sym}^k\Lambda \otimes H^0(C,L)) \to \mathrm{Sym}^N Z\,,$$

it is surjective for large N, too. □

We now return to the proof of Theorem 1.4.2. It has been shown that, for large enough N, $H^0(C, L^{N(p+1)})$ has a basis of weight not exceeding

$$w_{0,0}(q_{0,0} - q_{0,1}) + \cdots + w_{l-1,p-1}(q_{l-1,p-1} - q_{l,0}) + N(p+1)\rho_s h^0(C, L^{N(p+1)}),$$

where

$$q_{j,k} = N(p-k)\rho_{h_j} + Nk\rho_{h_{j+1}} + N\rho_s$$

and $w_{j,k}$ is the dimension of the image of

$$\mathrm{Sym}^{N(p-k)}V_{h_j} \otimes \mathrm{Sym}^{Nk}V_{h_{j+1}} \otimes \mathrm{Sym}^N V \longrightarrow H^0(C, L^{N(p+1)})\,.$$

On the other hand, Lemma 1.4.3 shows that, for large N,

$$w_{j,k} \geq N(p-k)d_{h_j} + Nkd_{h_{j+1}}\,.$$

Combining everything, we find that for large N there is a basis of $H^0(C, L^{N(p+1)})$ of weight not greater than

$$\begin{aligned}
&-\sum_{j=0}^{l-1}\sum_{k=0}^{p-1} N^2\left((p-k)d_{h_j} + kd_{h_{j+1}}\right)\left(\rho_{h_{j+1}} - \rho_{h_j}\right) + N(p+1)\rho_s(N(p+1)d + 1 - g) = \\
&= -N^2\sum_{j=0}^{l-1}\left(\frac{p^2+p}{2}d_{h_j} + \frac{p^2-p}{2}d_{h_{j+1}}\right)\left(\rho_{h_{j+1}} - \rho_{h_j}\right) + N^2(p+1)^2\rho_s d + \\
&\quad + N(p+1)\rho_s(1-g) = \\
&= -N^2p^2\sum_{j=0}^{l-1}\frac{d_{h_j} + d_{h_{j+1}}}{2}\left(\rho_{h_{j+1}} - \rho_{h_j}\right) + N^2p\sum_{j=0}^{l-1}\left(d_{h_{j+1}} - d_{h_j}\right)\left(\rho_{h_{j+1}} - \rho_{h_j}\right) + \\
&\quad + N^2(p+1)^2\rho_s d + N(p+1)\rho_s(1-g)\,.
\end{aligned}$$

What has to be shown is that this quantity, which we denote by w, is negative for an appropriate choice of p and of $h_0, \dots, h_l$. We have not yet used the hypothesis that $C \subset \mathbb{P}^s$ be linearly stable. This implies, in particular, that $d_r > rd/s$ for $0 \le r < s$. Thus, if m_r stands for the minimum integer larger than rd/s, and we set

$$\varepsilon = \min\left\{ m_r - \frac{rd}{s} \mid r = 0, \dots, s-1 \right\} ,$$

we find that $\varepsilon > 0$ and that

$$d_r \ge \frac{rd}{s} + \varepsilon$$

for $0 \le r < s$. As a consequence

$$\sum_{j=0}^{l-1} \frac{d_{h_j} + d_{h_{j+1}}}{2} \left(\rho_{h_{j+1}} - \rho_{h_j}\right) \ge \frac{d}{s} \sum_{j=0}^{l-1} \frac{h_j + h_{j+1}}{2} \left(\rho_{h_{j+1}} - \rho_{h_j}\right) + \varepsilon\left(\rho_s - \rho_0\right) ,$$

so that

$$\begin{aligned} w \le -N^2p^2\frac{d}{s} \sum_{j=0}^{l-1} \frac{h_j + h_{j+1}}{2} \left(\rho_{h_{j+1}} - \rho_{h_j}\right) - N^2p^2\varepsilon\left(\rho_s - \rho_0\right) \\ + N^2pd(\rho_s - \rho_0) + N^2(p+1)^2\rho_s d \,. \end{aligned} \tag{1.8}$$

The time has come to choose p and $h_0, \dots, h_l$. The correct choice is provided by the following combinatorial result (cf. [Mum77], Lemma 4.13).

Lemma 1.4.4. *Let $\rho_0 \le \rho_1 \le \cdots \le \rho_s$ be real numbers. Then*

$$\max_{l,r} \left(\sum_{j=0}^{l-1} \frac{h_j + h_{j+1}}{2} \left(\rho_{h_{j+1}} - \rho_{h_j}\right) \right) \ge s\rho_s - \frac{s}{s+1} \sum_i \rho_i \tag{1.9}$$

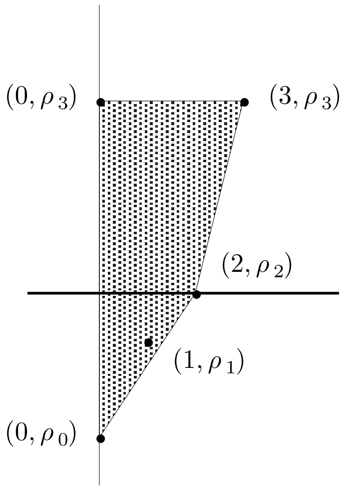
where l varies among all integers between 1 and s and r among all sequences of integers $0 = h_0 < h_1 < \cdots < h_l = s$.

Before proving the lemma, we use it to conclude the proof of Theorem 1.4.2. Pick a sequence $0 = h_0 < h_1 < \cdots < h_l = s$ for which the maximum in the left-hand side of (1.9) is attained. In our situation, $\sum \rho_i = 0$ and $\rho_0 < 0$, so (1.9) and (1.8) yield

$$\begin{aligned} w &\le -N^2p^2d\rho_s - N^2p^2\varepsilon\left(\rho_s - \rho_0\right) + N^2pd(\rho_s - \rho_0) + N^2(p+1)^2\rho_s d \\ &= -N^2p(p\varepsilon - d)(\rho_s - \rho_0) + 2N^2pd\rho_s + N^2d\rho_s \\ &\le -N^2p(p\varepsilon - d)(\rho_s - \rho_0) + 2N^2pd(\rho_s - \rho_0) + N^2d(\rho_s - \rho_0) \\ &= -N^2(\rho_s - \rho_0)[p(p\varepsilon - d) - 2pd - d] \,. \end{aligned}$$

We choose a p so large that $p(p\varepsilon - d) - 2pd - d > 0$. Since $H^0(C, L^{N(p+1)})$ has a basis of weight at most w, it then has a basis of negative weight. This shows that the h-th Hilbert point of $C \subset \mathbb{P}^s$ is stable for $h = N(p+1)$ and N sufficiently large.

To conclude the proof of Theorem 1.4.2 it remains to prove Lemma 1.4.4. The left-hand side of (1.9) is just the area of the convex hull A of the points $(0, \rho_s)$, $(0, \rho_0)$, $(1, \rho_1)$,..., (s, ρ_s) in the plane (the picture below shows an example with $s = 3$).



Decreasing the ρ_i with $0 < i < s$ so as to bring the points (i, ρ_i) down onto the boundary of A obviously does not change the left-hand side of (1.9), while it increases the right-hand side. Therefore we may limit ourselves to proving (1.9) in the special case when all the points (i, ρ_i) lie on the lower boundary of A. In this case the area of A is just

$$\sum_{j=0}^{s-1} \frac{2j+1}{2}(\rho_{j+1} - \rho_j)\,.$$

But this quantity equals

$$\sum_{j=1}^{s} j(\rho_j - \rho_{j-1}) - \frac{1}{2}(\rho_s - \rho_0) = s\rho_s + \frac{1}{2}(\rho_s + \rho_0) - \sum_i \rho_i\,.$$

On the other hand

$$s\rho_s + \frac{1}{2}(\rho_s + \rho_0) - \sum_i \rho_i = s\rho_s + \frac{1}{2}(\rho_s + \rho_0) - \frac{1}{s+1}\sum_i \rho_i - \frac{s}{s+1}\sum_i \rho_i \geq s\rho_s - \frac{s}{s+1}\sum_i \rho_i$$

since A is convex. This finishes the proof of Lemma 1.4.4, and hence of Theorem 1.4.2. □

Remark 1.4.5. By Remark 1.1.6 this Theorem implies that for *any* morphism ψ from an irreducible curve to a projective space, linear stability implies generalised Hilbert stability.

Remark 1.4.6. There is at least another non-direct way of proving Theorem 1.5.1, passing through another concept of stability, the (asymptotical) Chow stability (see [Mum77]). Indeed, it is shown in [Mum77] (Theorem 4.12) that for curves embedded in projective space linear (semi-)stability implies Chow (semi-)stability. On the other hand, from [Mor80], corollary 3.5 (i), we learn that Chow stability implies Hilbert stability. However, Morrison shows as well that in the case of semi-stability, the last implication is reversed, i.e.

$$\text{Hilbert semi-stability} \implies \text{Chow semi-stability.}$$

Also the argument of Gieseker used in Theorem 1.4.2 makes a crucial use of the linear stability and doesn't work under the weaker hypothesis of linear semi-stability. It is not known whether linear semi-stability implies Hilbert semi-stability or not.

1.5 The Cornalba-Harris Theorem generalised

In this section we prove a generalised version of Theorem (1.1) of [CH88]. This will be our key tool for proving inequalities for the slope of fibred surfaces in chapter 2.

The idea of the theorem is the following. Consider a family of algebraic varieties X_t over a base T and a family L_t of line bundles on it. Under the assumption that for general $t \in T$ the map induced by $|L_t|$ is semi-stable in the sense defined in section 1.1, this theorem constructs a line bundle on T together with a non-zero section. In particular, when T is a curve, this gives a non-trivial inequality involving the degrees of certain naturally defined rational classes of divisors on it.

Before stating and proving the theorem, we make some remarks on vector bundles and representations which will be needed in the proof.

Let T be a smooth complex projective variety. Consider a vector bundle E of rank r on T and a complex holomorphic representation of $GL(r, \mathbb{C})$

$$GL(r, \mathbb{C}) \xrightarrow{\rho} GL(V).$$

Composing the transition functions of E with ρ, we can construct a new vector bundle which we call E_ρ. To be more precise, if $\{g_{\alpha,\beta}\}$ is a system of transition functions for E with respect to an open cover $\{\mathcal{U}_{\alpha,\beta}\}$ of T, then a system of transition functions for E_ρ with respect to the same cover is $\{\rho(g_{\alpha,\beta})\}$. Clearly E_ρ has typical fibre V and structure group $\rho(GL(r, \mathbb{C}))$.

For instance, if we consider as ρ the representations corresponding to symmetric power, tensor power and exterior power,

$$GL(r, \mathbb{C}) \to GL(\mathrm{Sym}^n \mathbb{C}^r), \quad GL(r, \mathbb{C}) \to GL(\otimes^n \mathbb{C}^r), \quad GL(r, \mathbb{C}) \to GL(\wedge^n \mathbb{C}^r)$$

we obtain as E_ρ respectively the vector bundles $\mathrm{Sym}^n E$, $\otimes^n E$ and $\wedge^n E$.

Suppose now that H is a line bundle on T and that we are given a bundle homomorphism

$$E_\rho \xrightarrow{\vartheta} H$$

and a $GL(r, \mathbb{C})$-invariant subspace $W \subseteq \mathrm{Sym}^l V$, and let $\sigma : GL(r, \mathbb{C}) \longrightarrow GL(W)$ be the representation obtained by restriction. In other words, $\{\mathcal{U}_{\alpha,\beta}\}$ is an open cover of T and $g_{\alpha,\beta}$ are the corresponding transition functions for E, the transition functions of E_σ, are the restrictions of the transition functions $\mathrm{Sym}^l \rho(g_{\alpha,\beta})$ of $\mathrm{Sym}^l E_\rho$.

Thus, there is an inclusion of vector bundles $E_\sigma \hookrightarrow \mathrm{Sym}^l E_\rho$. Composing this inclusion with $\mathrm{Sym}^l \vartheta$, we obtain a homomorphism

$$E_\sigma \xrightarrow{\Theta} H^l.$$

1.5.1 The theorem

The argument below is an adaptation of the proof of the Cornalba-Harris theorem given in chapter 14 of [ACGH]. Given a sheaf $\mathcal{F}$ over a variety T, we call $\mathcal{F} \otimes \boldsymbol{k}(t)$ the fibre of $\mathcal{F}$ over the point $t \in T$.

Theorem 1.5.1. *Let $f : X \to T$ be a flat proper morphism from a variety X to a smooth projective variety T. Let t be a general point of T, X_t the fibre of f at t. Let L be a line bundle on X and $\mathcal{F}$ a locally free subsheaf of f_*L of rank r. Consider the linear system*

$$\mathcal{F} \otimes \boldsymbol{k}(t) \subseteq H^0(X_t, L_{|X_t}).$$

Suppose that there is an integer $h \geq 0$ (not depending on t) such that the associated generalised h-th Hilbert point is GIT semi-stable with respect to the $SL(r, \mathbb{C})$-action described in section 1.1.

Let $\mathcal{G}_h \subseteq f_*L^h$ be a coherent subsheaf that contains the image of the morphism

$$\mathrm{Sym}^h \mathcal{F} \longrightarrow f_*L^h,$$

and coincides with it at t. Set $N = N(h) = \operatorname{rank} \mathcal{G}_h$. Let $\mathcal{L}_h$ be the line bundle

$$\mathcal{L}_h = \det(\mathcal{G}_h)^r \otimes (\det \mathcal{F})^{-h\,N}.$$

Then there is a positive integer m, depending only on h, $\operatorname{rank}\mathcal{F}$ and $\operatorname{rank}\mathcal{G}_h$, such that $(\mathcal{L}_h)^m$ is effective. In particular, if T is a curve, the following inequality holds:

$$r \deg \mathcal{G}_h - h\,N \deg \mathcal{F} \geq 0.$$

Proof. In what follows, t is a general point of T. We want to show that there is a positive integer m, depending only on h, r and N, such that the line bundle $\mathcal{L}_h^m$ has a nonzero section. Consider the morphism

$$\mathrm{Sym}^h \mathcal{F} \xrightarrow{\gamma_h} \mathcal{G}_h.$$

Its fibre at t is

$$\mathrm{Sym}^h \mathcal{F} \otimes \boldsymbol{k}(t) \xrightarrow{\gamma_h \otimes \boldsymbol{k}(t)} \mathcal{G}_h \otimes \boldsymbol{k}(t),$$

which is surjective according to the assumption. The fibre of

$$\wedge^N \mathrm{Sym}^h \mathcal{F} \xrightarrow{\wedge^N \gamma_h} \det \mathcal{G}_h$$

at t is GIT semi-stable by assumption. Therefore there exists a homogeneous $SL(r, \mathbb{C})$-invariant polynomial

$$P \in \mathrm{Sym}(\wedge^N \mathrm{Sym}^h (\mathcal{F} \otimes \boldsymbol{k}(t)) \otimes \det(\mathcal{G}_h \otimes \boldsymbol{k}(t))^\vee)$$

not vanishing at $\wedge^N \gamma_h \otimes \boldsymbol{k}(t)$.

We may assume (simply taking a power of P if necessary) that the degree of P is mr, where m is an integer depending only on h, r and N.

Now the idea is to somehow "evaluate P on $\wedge^N \gamma_h$". The result will be the section we wish to construct. We will use the remarks on representations made above.

The section we wish to construct will be a special instance of Θ. We take $E = \mathcal{F}$, $H = \det \mathcal{G}_h$, and choose as ρ the N-th exterior power of the h-th symmetric power of the standard representation $\mu : GL(r, \mathbb{C}) \to GL(\mathbb{C}^r)$. Therefore

$$\rho : GL(r, \mathbb{C}) \to GL(\wedge^N \mathrm{Sym}^h H^0(X_t, L_{|X_t})).$$

With these choices $E_\rho = \wedge^N \mathrm{Sym}^h \mathcal{F}$, and we may take as ϑ the homomorphism $\wedge^N \gamma_h$. We can fix isomorphisms $\mathcal{F} \otimes \boldsymbol{k}(t) \cong \mathbb{C}^r$ and $\det G_h \otimes \boldsymbol{k}(t) \cong \mathbb{C}$, and consider P as an element of $\mathrm{Sym}^{mr}\mathbb{C}^r$. As this polynomial is $SL(r, \mathbb{C})$-invariant, changing the isomorphisms has the effect of multiplying it by a nonzero constant. We therefore choose as W the line in $\mathrm{Sym}^{mr}\mathbb{C}^r$ generated by P, which is independent of the isomorphisms chosen and $GL(r, \mathbb{C})$-invariant. More precisely, given an element $M \in GL(r, \mathbb{C})$, if we write $M = (\det M)^{1/r} U$, where $U \in SL(r, \mathbb{C})$, the action of M on P is the following:

$$\sigma(M)P = \mathrm{Sym}^{mr}\rho((\det M)^{1/r}U)P = \det \mu(M)^{hNm} \mathrm{Sym}^{mr}\rho(U)P = \det \mu(M)^{hNm} P.$$

It follows that in our case E_σ is the line bundle $(\det \mathcal{F})^{hNm}$ and $\Theta : E_\sigma \to H^{mr}$ is the composite homomorphism

$$\det \mathcal{F}^{hNm} \hookrightarrow \mathrm{Sym}^{mr}(\wedge^N \mathrm{Sym}^h \mathcal{F}) \to (\det \mathcal{G}_h)^{mr}.$$

The fibre of $E_\sigma = \det \mathcal{F}^{hNm}$ at t is the line generated by P. Hence,the fact that the homomorphism Θ is nonzero at t is equivalent to the non-vanishing of P at $\wedge^N \gamma_h \otimes \boldsymbol{k}(t)$.

The proof of Theorem 1.5.1 is therefore concluded. □

Remark 1.5.2. Theorem (1.1) of [CH88] requires the sheaf $\mathcal{F}$ to give a semi-stable *embedding* on the general fibre. But in fact the proof works with almost no changes, as semi-stability is the crucial hypothesis.

This generalisation sounds a little unnatural because, as GIT is mainly used to construct moduli spaces, GIT stability is usually defined for line bundles whose associated morphisms encode all the informations about the variety, as in the case of the Hilbert point of a smooth curve. But the method of Cornalba and Harris does not need all this, and, as we shall see in the next chapter, gives interesting inequalities even with the weaker assumption we introduced.

There is a version of this theorem over positive-characteristic fields, due to Bost (cf [Bos94]). It is worth noticing that Bost's version is itself a generalisation of Theorem (1.1) of [CH88]. Indeed, it requires only the weaker assumption of Chow (semi-)stability of the morphism induced on the general fibres, instead of Hilbert (semi-)stability.

1.5.2 Applications to the effective cone of T

In all the applications of Theorem 1.5.1 that have been made so far, ours not being exceptions, the condition of stability is satisfied not for a *fixed* h, but for *h large enough.* More precisely Hilbert stability of the map induced on the general fibre is satisfied. In this paragraph, we want to give as much as possible explicit computations of the classes of line bundles involved in the theorem of Cornalba and Harris, and we draw some conclusions on the effective cone of the base T (cf. Theorem (1.1) and Corollary (1.2) of [CH88]).

Suppose that the assumptions of Theorem 1.5.1 are satisfied, for infinitely many positive integers h. Let k be the dimension of T and $d+k$ the dimension of X (so f has relative dimension d). Theorem 1.5.1 assures that for infinitely many positive integers h there exist an integer m such that the line bundle

$$\mathcal{L}_h^m = \left(\det(\mathcal{G}_h)^r \otimes (\det \mathcal{F})^{-hN}\right)^m$$

is effective. Its first rational Chern class $c_1(\mathcal{L}_h) \in A^1(T)_{\mathbb{Q}}$ is a polynomial in h of degree $d+1$.

$$c_1(\mathcal{L}_h) = \alpha_{d+1}h^{d+1} + \alpha_d h^d + \ldots + \alpha_0, \quad \alpha_i \in A^1(T)_{\mathbb{Q}}.$$

If m were independent of h, we could easily conclude that the leading coefficient α_{d+1} of this polynomial is the limit in $A^1(T)_{\mathbb{Q}}$ of effective divisors, i.e. it lies in the closure of the effective cone $\mathrm{Eff}_{\mathbb{Q}}(T)$. This not being the case, we may argue as follows (cf. [CH88], Theorem (1.1)). If E is any effective divisor class on T,

$$\alpha_{d+1} = \frac{E + \frac{c_1(\mathcal{L}_h^m)}{m}}{h^{d+1}} + \frac{R(h)}{h^{d+1}},$$

where R is a polynomial of degree at most d. Since the divisor class $E + c_1(\mathcal{L}_h^m)/m = E + c_1(\mathcal{L}_h)$ is effective, letting h go to infinity we see that α_{d+1} is indeed the limit of effective divisor classes.

Some intersection theoretical computations

We can make explicit computations and simplifications under some additional assumptions.

Assumption 1.5.3. *Suppose that*

- *the sheaf $\mathcal{G}_h$ generically coincides with f_*L^h.*[2]
- *X is smooth.* [3]

This assumed, remembering that $\mathcal{G}_h \subseteq f_*L$ and the quotient is torsion, we have that $c_1(f_*L) \geq c_1(\mathcal{G}_h)$, so

$$c_1(\mathcal{L}_h) \leq r\, c_1(f_*L^h) - h\, \mathrm{rank} f_*L^h\, c_1(\mathcal{F}).$$

By the Grothendieck-Riemann-Roch Theorem ([Ful84], Theorem 15.2), the following equality holds in $A_*(T)_{\mathbb{Q}}$.

$$\mathrm{ch}\left(f_! L^h \cap \mathrm{td}(\mathcal{O}_T)\right) = f_*\left(\mathrm{ch}(L^h) \cap \mathrm{td}(\mathcal{O}_X)\right), \tag{1.10}$$

where

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rank}\mathcal{E}\,[X] + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \ldots \in A_*(X)_{\mathbb{Q}}$$

[2]Note that this condition is satisfied if we assume that the general fibres of the sheaf $\mathcal{F}$ induce embeddings on the fibres.

[3]This assumption can be relaxed using the version of the Grothendieck-Riemann-Roch Theorem for singular varieties (cf. [BFM79] and [Ful84]).

and

$$\mathrm{td}(\mathcal{E}) = [X] + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}(c_1(\mathcal{E})^2 + c_2(\mathcal{E})) + \ldots \in A_*(X)_{\mathbb{Q}}$$

are respectively the Chern character and the Todd class of a sheaf $\mathcal{E}$ on X, and

$$f_!\mathcal{E} = \sum_i (-1)^i R^i f_* \mathcal{E}.$$

Moreover, recall that, as L is a line bundle,

$$\mathrm{ch}(L^h) = \sum_{i=0}^{k+d} \frac{c_1(L^h)^i}{i!} = \sum_{i=0}^{k+d} \frac{h^i c_1(L)^i}{i!}.$$

Let us consider the a-codimensional part of equality (1.10),

$$\left[\mathrm{ch}\left(f_! L^h \cap \mathrm{td}(\mathcal{O}_T)\right)\right]^a = f_*\left(\left[\mathrm{ch}(L^h)\right]^{d+a} \cap \mathrm{td}(\mathcal{O}_X)\right).$$

Hence, putting $a = 1$, we get

$$c_1(f_! L^h) \cap [T] = \frac{h^{d+1}}{(d+1)!} f_*\left(c_1(L)^{d+1} \cap [X]\right) + O(h^d),$$

and for $a = 0$

$$\mathrm{rank} f_! L^h \cdot [T] = \frac{h^d}{d!} f_*\left(c_1(L)^d \cap [X]\right) + O(h^{d-1}).$$

Putting everything together, we obtain

$$\begin{aligned} c_1(\mathcal{L}_h) \cap [T] \leq \;& \tfrac{h^{d+1}}{(d+1)!}\left(r f_*(c_1(L)^{d+1} \cap [X]) - (d+1)c_1(\mathcal{F}) \cap f_*(c_1(L)^d \cap [X])\right) + \\ & + \textstyle\sum_{i=1}^{d} (-1)^{i+1}\left(r c_1(R^i f_* L^h) \cap [T] - h\,\mathrm{rank}(R^i f_* L^h)\, c_1(\mathcal{F}) \cap [T]\right) + \\ & + O(h^d). \end{aligned} \tag{1.11}$$

Let us make the further assumption that L is *relatively f-ample*, i.e. that for any coherent sheaf $\mathcal{E}$ on X, the canonical morphism

$$f^* f_*(\mathcal{E} \otimes L^n) \longrightarrow \mathcal{E} \otimes L^n$$

is surjective for all large enough n. This is equivalent to the assumption that the restriction of L to *every* fibre of f is an ample line bundle (because f is proper, cf. [Deb01], 7.41). Under this assumption the higher direct images of $f_* L^h$ vanish for $h >> 0$, as shown for instance in [Har77] (Theorem III.8.8). Hence, the inequality reads simply

$$c_1(\mathcal{L}_h) \cap [T] \leq \frac{h^{d+1}}{(d+1)!}\left(r f_*(c_1(L)^{d+1} \cap [X]) - (d+1)c_1(\mathcal{F}) \cap f_*(c_1(L)^d \cap [X])\right) + O(h^d).$$

If we call $\mathcal{E}(L, \mathcal{F})$ the class $r f_*(c_1(L)^{d+1} \cap [X]) - (d+1)c_1(\mathcal{F}) \cap f_*(c_1(L)^d \cap [X])$, as in [CH88] and in [Bos94], we can conclude that $\mathcal{E}(L, \mathcal{F})$ is contained in the closure of the effective cone of $A_{k-1}(T)_{\mathbb{Q}}$. This is the conclusion of the original Cornalba-Harris Theorem in [CH88].

Inequalities for fibred surfaces

Suppose now that X is a surface and $T = B$ is a curve. In this case we call $f\colon X \to B$ a fibred surface (see Definition 2.0.5). The classes of invariants of fibred surfaces are the object of our study in Chapter 2, and Theorem 1.5.1 will be our main tool.

In this case, fixing any isomorphism $A^1(B) \cong \mathbb{Z}$ the number $c_1(\mathcal{L}_h)$ coincides with $\deg \mathcal{L}_h$. It can be seen again as a polynomial in $\mathbb{Q}[h]$ of degree 2, and Theorem 1.5.1 says it is non-negative for h large enough. As above, we consider its leading coefficient, which has to be greater or equal than 0. In particular, we can state the following consequence of Theorem 1.5.1:

Corollary 1.5.4. *Let $f\colon X \to B$ be a fibred surface. Let L be a line bundle on X and $\mathcal{F}$ a locally free coherent subsheaf of f_*L of rank r such that for general $b \in B$ the linear system*

$$\mathcal{F} \otimes \boldsymbol{k}(b) \subseteq H^0(X_b, L_{|X_b})$$

induces a Hilbert semi-stable rational map (Definition 1.1.7). Let $\mathcal{G}_h$ be a coherent subsheaf of f_*L^h that contains the image of the morphism

$$\mathrm{Sym}^h \mathcal{F} \longrightarrow f_*L^h,$$

and coincides with it at general b. If $N = \mathrm{rank}\mathcal{G}_h$ is of the form $Ah + O(1)$ and $\deg \mathcal{G}_h$ of the form $Bh^2 + O(h)$, the following inequality holds:

$$rB - A \deg \mathcal{F} \geq 0. \tag{1.12}$$

Let us consider the particular case in which $\mathcal{F} = f_*L$ and $\mathcal{G}_h = f_*L^h$. By the Riemann-Roch Theorem

$$\deg f_*L^h = \deg f_!L^h + \deg R^1 f_*L^h = \frac{h^2}{2}(L \cdot L) - \frac{h}{2}(L \cdot \omega_f) + \deg f_*\omega_f + \deg R^1 f_*L^h.$$

Let d be the relative degree of L. For large enough h, By Riemann-Roch on the general fibre, $N = dh - g + 1$, where g is the genus of the fibration. Suppose that $\deg R^1 f_*L^h = Ch^2 + O(h)$; in this case the computation of the leading coefficient of $\deg \mathcal{L}_h$ gives:

$$r(L \cdot L) + r\,C - 2d \deg f_*L \geq 0. \tag{1.13}$$

Chapter 2

The slope of fibred surfaces

Here we recall some definitions and basic results about fibred surfaces and their invariants.

Definition 2.0.5. *We define a* fibred surface *(or* fibration*, for short) to be the data of a smooth projective surface X with a proper surjective morphism with connected fibres f to a smooth complete curve B.*

Observe that such a morphism is automatically flat, in fact under the assumptions of the definition flatness is *equivalent* to surjectivity (see [Har77] Prop.9.7.III). Moreover, the generic fibres of a fibration are *smooth* (see for instance [Bad00] pag.90 prop.7.4). The genus g of the general fibres (which, by what observed above, coincides with the arithmetic genus of the singular ones), is called the genus of the fibration.

Define a (-1)-curve (respectively a (-2)-curve) to be a nonsingular rational curve $C \subset X$ with self-intersection -1 (respectively -2).

Definition 2.0.6. *Let $f\colon X \to B$ be a fibration of positive genus. We call f* relatively minimal *if there are no (-1)-curves contained in the fibres.*

Given a fibration $f : X \to B$, let $\overline{X}$ be the surface obtained from X contracting the (-1)-curves contained in the fibres. As it is well-known, $\overline{X}$ is smooth and has an induced fibration on B, called the relatively minimal model of f.

From now on we will always consider *relatively minimal fibrations of genus $g \geq 2$.*

Definition 2.0.7. *A fibration is said to be* semi-stable *if all the fibres are moduli semi-stable curves, that's to say reduced, with only nodes as singularities and not containing (-1)-curves.*

Definition 2.0.8. *We say that a fibration is* smooth *if all its fibres are smooth curves;* isotrivial *if all its smooth fibres are mutually isomorphic;* locally trivial *if it is smooth and isotrivial.*

The Grauert-Fisher Theorem (cf. [BHPdV04] Theorem I.10.1) assures that the last definition coincides with the standard definition of local triviality, (i.e. that the fibration is a holomorphic fibre bundle).

An isotrivial fibration is birationally isomorphic to a quotient of a product of curves by the action of a finite group (cf. [Ser96]).

Definition 2.0.9. *We say that a fibration is k-gonal if the general fibres have gonality k. In particular, a fibration is hyperelliptic (trigonal, tetragonal) if the general fibres are.*

Albanese fibrations

Recall that the irregularity of a surface X is the integer

$$q = q(X) = h^{0,1}(X) = h^1(X, \mathcal{O}_X) = h^1(X, K_X).$$

Let $f : X \to B$ be a fibred surface. Let $X_t \overset{i_t}{\hookrightarrow} X$ be a smooth fibre over $t \in B$, and consider the commutative diagram induced by the universal property of the Albanese maps

$$\begin{array}{ccc} X_t & \xrightarrow{alb_{X_t}} & J(X_t) \\ {\scriptstyle i_t}\downarrow & & \downarrow{\scriptstyle (i_t)_*} \\ X & \xrightarrow{alb_X} & Alb(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ B & \xrightarrow{alb_B} & J(B) \end{array}$$

where $J(C)$ indicates the Jacobian of a curve C. Notice that f_* is surjective, because f is. The abelian variety $Alb(X)$, as any complex torus, is rigid; therefore, the subvariety

$$(i_t)_*(J(X_t)) \subset Alb(X)$$

does not depend on t, so we call it simply A. Call b the genus of the base B. From what observed above we get

$$q = b + \dim A \leq b + g.$$

Definition 2.0.10. *We call a fibred surface $f : X \to B$ an Albanese fibration if $q = b$.*

If $b \geq 1$, $alb_B(B) \cong B$, and from the diagram and the universal property of Albanese varieties we see that $b = q$ if and only if $alb_X(X) \cong B$. Notice that the Albanese fibrations are precisely the fibrations for which f is exactly alb_X when restricting its image.

On the other hand, $q = b + g$ if and only if f is a trivial fibration, i.e. $X \cong B \times F$ and f is the first projection into B (cf. [Bea82]).

Relative canonical sheaf

Given a smooth variety X of dimension n, let $\omega_X = \Omega^n_X$ be its canonical line bundle.

Let $f : X \to B$ be a fibred surface.

Definition 2.0.11. *The line bundle $\omega_f = \omega_X \otimes f^*(\omega_B)^{-1}$ is called the relative canonical sheaf.*

Recall that for any fibre F of a fibration the normal bundle $\mathcal{O}_F(F)$ is trivial (cf. [BHPdV04] Lemma III.8.1). Suppose that F is smooth. By the adjunction formula

$$\omega_{f|F} = (\omega_X \otimes f^*(\omega_B)^{-1})_{|F} \cong \omega_X(F)_{|F} \cong \omega_F.$$

If F is singular (possibly non-reduced) we formally define the *dualizing bundle* on F as

$$\omega_F := \omega_X \otimes \mathcal{O}_F(F).$$

It is a locally free sheaf, and its name derives from the fact that it satisfies the following *duality property.* There exists a trace homomorphism $tr : H^1(F, \omega_F) \to \mathbb{C}$ such that, for any coherent sheaf $\mathcal{F}$ over F, the pairing

$$Hom\,(\mathcal{F}, \omega_F) \times H^1(F, \mathcal{F}) \longrightarrow H^1(F, \omega_F) \xrightarrow{tr} \mathbb{C},$$

induces an isomorphism $Hom\,(\mathcal{F}, \omega_F) \cong H^1(F, \mathcal{F})^\vee$.

The pushforward of the dualizing sheaf $f_*\omega_f$ is a locally free sheaf [1] on B of rank g.

Relative invariants

The basic invariants for a relatively minimal fibration $f\colon X \to B$ are:

$$(\omega_f \cdot \omega_f) \;,\quad \deg f_*\omega_f \quad \text{and} \quad e_f := e(X) - e(B)e(F),$$

where $e(\ )$ is the topological Euler number, i.e. the top Chern class of the variety.

Using Leray's spectral sequence, it can be shown that $\deg f_*\omega_f$ coincides with the relative Euler characteristic $\chi_f = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_B)\chi(\mathcal{O}_F)$, where

$$\chi(\mathcal{O}_X) = \sum (-1)^i h^i(X, \mathcal{O}_X).$$

Notice that the relative invariants are connected to the invariants of the surface X by the following formulas

$$\begin{aligned}(\omega_f \cdot \omega_f) &= (\omega_X \cdot \omega_X) - 8(b-1)(g-1),\\ \deg f_*\omega_f &= \chi(\mathcal{O}_X) - (b-1)(g-1),\\ e_f &= e(X) - 4(b-1)(g-1),\end{aligned}$$

where b is the genus of the base B. Moreover, they are related by Noether's formula (which is a consequence of the Grothendieck-Riemann-Roch Theorem):

$$(\omega_f \cdot \omega_f) = 12 \deg f_*\omega_f - e_f.$$

It is well known that all these three basic invariants are non-negative (cf, [Bea82]). Moreover,

- $\deg f_*\omega_f = 0$ if and only if f is locally trivial (Theorem III.18.2 of [BHPdV04]);
- $(\omega_f \cdot \omega_f) = 0$ implies that f is isotrivial ([Ara71]);
- $e_f = 0$ if and only if the fibration is smooth ([Bea78]).

[1] More generally, given a fibration $f : X \to B$, the pushforward of any locally free sheaf $\mathcal{F}$ on X is again a locally free sheaf on B. Indeed, $f_*\mathcal{F}$ is torsion free (all we need for this is the surjectivity of f). On the other hand, for any $b \in B$, the stalk $\mathcal{O}_{B,b}$ is a discrete valuation ring, because B is smooth of dimension one. So the stalk $(f_*\mathcal{F})_b$ is a non-torsion finitely generated module on a principal ideal domain. Hence, the structure theorem for modules over PID (cf. for instance [Jac85] 3.8) tells us that $(f_*\mathcal{F})_b$ has to be locally free.

Slope

Assuming that the fibration is not locally trivial, we can consider the ratio

$$s(f) = \frac{(\omega_f \cdot \omega_f)}{\deg f_*\omega_f},$$

which is itself an important invariant of the fibration, called the *slope*.

2.1 Lower bounds for the slope

Noether's formula gives the upper bound for the slope $s(f) \leq 12$, which is achieved when all the fibres are smooth (e.g. for the Kodaira fibrations).

The search for a sharp lower bound has been more difficult. The bound is given by the following inequality, which we call *slope inequality*:

$$(\omega_f \cdot \omega_f) \geq \frac{4(g-1)}{g} \deg f_*\omega_f; \tag{2.1}$$

that is, $s(f) \geq 4(g-1)/g$. This inequality was originally discovered by Horikawa and Persson ([Hor81] [Per82]) for hyperelliptic fibrations; for arbitrary fibrations it was independently proved in the eighties by Xiao ([Xia87a]) and by Cornalba-Harris ([CH88]). However, Cornalba and Harris dealt only with semi-stable fibrations, as their interest was in the applications to the moduli space of stable curves $\overline{M}_g$. In particular, they derived the slope for non-hyperelliptic semi-stable relatively minimal fibrations as a corollary from a more general result (Theorem (1.1) of [CH88]). On the other hand, the hyperelliptic semi-stable case was obtained by an ad hoc argument, relying on an identity in the rational Picard group of the hyperelliptic locus of $\overline{M}_g$, also proved in [CH88]. It should be remarked that the inequality for semi-stable fibrations doesn't imply the inequality in general, as we point out in 2.1.1.

Observe that the slope inequality implies in particular that $(\omega_f \cdot \omega_f) = 0$ if and only if f is locally trivial. The inequality is sharp, and it is possible to classify the fibrations that reach it, which are in particular all hyperelliptic (see [Kon93] and [AK00] or [CH88] for the semi-stable case). In Example 2.3.11, in particular, we exhibit a family of hyperelliptic fibrations of arbitrary genus g with slope $4(g-1)/g$.

2.1.1 Why semi-stable fibrations are not enough

As is well-known, the process of *semi-stable reduction* (cf.[BHPdV04] Theorem III.10.3) associates to any fibred surface a *semi-stable* one, by means of a ramified base change. However, as Tan has shown in [Tan94] and [Tan96], the behaviour of the slope under base change *cannot be controlled when the base change ramifies over fibres which are not moduli semi-stable*, which is precisely what happens in the semi-stable reduction process. More precisely, the part of Tan's result that matters in this context can be stated as follows (cf. Theorem A and Theorem B of [Tan96]).

Theorem 2.1.1 (Tan)**.** *Let $f\colon X \to B$ be a relatively minimal fibration which is not locally trivial. Let $\pi\colon \widetilde{B} \to B$ be a morphism such that the relative minimal model $\widetilde{X}$ of the fibre* *product* $X \times_B \widetilde{B}$ *is a semi-stable fibration* $\widetilde{f}\colon \widetilde{X} \to \widetilde{B}$*. Let* $R_\pi \subseteq B$ *be the ramification divisor of* π*, and set* $\mathcal{R}_\pi = f^*(R_\pi)$*. Then*

$$s(\widetilde{f}) = \frac{(\omega_f \cdot \omega_f) - c_1^2(\mathcal{R}_\pi)}{\deg f_*\omega_f - \chi(\mathcal{O}_{\mathcal{R}_\pi})}. \tag{2.2}$$

The numbers $c_1^2(\mathcal{R}_\pi)$ *and* $\chi(\mathcal{O}_{\mathcal{R}_\pi})$ *are non-negative, are equal to* 0 *if and only if* f *is semi-stable, and satisfy the inequality:*

$$c_1^2(\mathcal{R}_\pi) \leq 8\chi(\mathcal{O}_{\mathcal{R}_\pi}).$$

Equality holds if and only if all the non-semi-stable fibres of f *are multiples of nodal curves.*

From this result we can derive that, if $s(\widetilde{f}) \geq 8$, then $s(f) \geq s(\widetilde{f})$. On the other hand, if

$$s(\widetilde{f}) < \frac{c_1^2(\mathcal{R}_\pi)}{\chi(\mathcal{O}_{\mathcal{R}_\pi})},$$

then $s(f) < s(\widetilde{f})$.

Hence, base change is a forbidden operation when one wants to prove lower bounds on the slope of non-semi-stable fibred surfaces. In particular, the inequalities that can be shown to hold for semi-stable fibrations, do not necessarily extend to arbitrary fibrations.

We have the same kind of problem in section 2.3.9, when we deal with double cover fibrations. Indeed, we will have to exclude those cases in which a base change is needed in order to prove the bound on the slope.

2.1.2 Overview of known results

One of the main problems in the study of fibred surfaces is to understand how *properties of the general fibre* F *influence the slope*; in particular considerable attention has been given to the problem of how the lower bound increases depending on the properties of the general fibres.

From this point of view, it is significant that, if the bound $4(g-1)/g$ is reached, then F has a g^1_2. As a matter of fact, the *gonality* (or the *Clifford index*) of the general fibre plays an important role with respect to the lower bound of the slope, as is apparent from the known results, of which we give here a brief account.

Trigonal fibrations

A first natural question is to find a lower bound for the slope of *trigonal* fibrations. The main results are the following.

Konno ([Kon96]) If $f : X \to B$ *is a trigonal fibration of genus* $g \geq 6$*, then*

$$s(f) \geq \frac{14(g-1)}{3g+1}. \tag{2.3}$$

Stankova-Frenkel ([SF00]) If $f : X \to B$ is a trigonal semi-stable fibration, then

$$s(f) \geq \frac{24(g-1)}{5g+1}. \tag{2.4}$$

Moreover, Stankova-Frenkel shows that the bound (2.4) is sharp and gives equivalent conditions for it to be achieved.

Konno proves in [Kon99] (corollary 4.4) that if $f_*\omega_f$ is a semi-stable vector bundle, and the fibration is non-hyperelliptic, then the better bound

$$s(f) \geq \frac{5g-6}{g} \tag{2.5}$$

holds. The same bound has been proved by Stankova-Frenkel for semi-stable trigonal fibrations satisfying some genericity property (cf. prop. 9.2 and prop. 12.3 of [SF00]).

Remark 2.1.2. Notice that the bound (2.5) is strictly greater than the bound (2.4) for $g \geq 4$. Hence, from the above results we can derive that if $f : X \to B$ is a (semi-stable) fibration of genus greater or equal than 4 whose slope reaches the bound (2.4), then $f_*\omega_f$ is an unstable vector bundle.

Almost nothing is known for higher gonality (apart from the results for general gonality exposed below). The natural guess would be that there exists a lower bound increasing with the gonality. In section 13 of [SF00], Stankova-Frenkel gives some partial results for tetragonal fibrations, and states some conjectures on the possible bounds. What seems clear is that a distinction has to be made between the general k-gonal case and the particular ones. For "general" we means the case of fibrations whose general fibres correspond to general points of the loci of k-gonal curves in the moduli space of curves $\overline{M}_g$ (see also Remark 2.1.3).

Influence of the Clifford index of the general fibre

In [Kon99], Konno obtains a bound for the slope depending on the Clifford index (see Definition 1.3.11) of the general fibres. However, his formula contains a term which is not easily computed. Anyway, it implies the bound (2.5) if the general fibre has Clifford index 1, and the bound

$$s(f) \geq \frac{5(g-1)}{g}$$

if the general fibre has Clifford index 2, under the additional assumption that $f_*\omega_f$ is a semi-stable vector bundle (corollary 4.4). Konno also proves that both these bounds are sharp (see example 4.6).

Double cover fibrations

A direction in which the hyperelliptic case can be generalised is the study of fibrations of genus g whose general fibre F is a double cover of a smooth curve of genus $\gamma \geq 0$, which we will call double fibrations of type (g, γ). As we will see in section 2.4, it is necessary to work with a smaller family of fibrations, i.e., fibrations that possess a *global* involution which restricts to an involution of the general fibres (double cover fibrations).

The known results about double cover fibrations are due to Barja and Zucconi.

Barja ([Bar01]) Let $f : X \to B$ is a bi-elliptic fibration of genus $g \geq 6$ (so f is of type $(g, 1)$). Then

$$s(f) \geq 4.$$

Moreover, $s(f) = 4$ if and only if X is the minimal desingularisation of a double cover $X_0 \to V$ of a smooth elliptic surface such that the elliptic fibration $V \to B$ is locally trivial and the branch divisor of the double cover has only negligeable singularities (see Definition 2.3.8).

Remark 2.1.3. Notice that a bi-elliptic fibration is a particular case of tetragonal fibrations. So from the sharpness of the bound we deduce that there are tetragonal fibrations of arbitrary genus with slope 4. As Barja observes in [Bar], since 4 is smaller than the slope (2.3) found by Konno for trigonal fibrations, this seem to contradict the idea of finding a bound for the slope as an increasing function of the gonality.

However, as observed above, the correct point of view seems to be to distinguish the general case, and indeed a bi-elliptic fibration is not general inside the locus of tetragonal curves.

Barja-Zucconi ([BZ01]) Let $f : X \to B$ be a double cover fibration of type (g, γ). Suppose that $g \geq 4\gamma + 1$. Then

$$s(f) \geq 4 + 4\frac{(\gamma - 1)(g - 4\gamma - 1)}{(g - 4\gamma - 1)(g - \gamma) + 2(g - 1)\gamma^2}.$$

Suppose that $g \geq 2\gamma + 11$ and that the general fibre of f is neither trigonal nor tetragonal. Then

$$s(f) \geq 4.$$

It is easy to construct examples of double cover fibrations with slope $4(g - 1)/(g - \gamma)$, while examples with smaller slope are known only for $g < 4\gamma$ (see [Bar] and Examples 2.3.11 and 2.3.12). Moreover, for hyperelliptic and bi-elliptic fibrations the number $4(g-1)/(g-\gamma)$ gives exactly the sharp bound. It is therefore natural to formulate the following

Conjecture 2.1.4 (Barja)**.** *Let $f : X \to B$ be a double cover fibration of type (g, γ) such that $g \geq 4\gamma + 1$. Then*

$$s(f) \geq 4\frac{g - 1}{g - \gamma}.$$

Bounds for "general" fibrations

As is apparent from the below results, it is natural to conjecture the sharp lower bound of the slope of "general " fibrations of genus g to be a function of g that approaches 6 from below as g grows. More precisely, here a "general" curve C means general in moduli, i.e. a smooth curve such that the corresponding point $[C]$ in the moduli space of stable curves $\overline{M}_g$ is contained in a Zariski open subset (for references on the moduli space of curves see for instance [HM98] and [ACGH]).

Eisenbud-Harris-Mumford ([HM82], [EH87], [Har84]) Let $f : X \to B$ be a semi-stable fibred surface such that the general fibre is general in moduli.

$$s(f) \geq 6 - o\left(\frac{1}{g}\right).$$

More precisely, if g is odd and B is not entirely contained in the divisor of k-gonal curves with $k \leq (g+1)/2$, then

$$s(f) \geq 6\frac{g-1}{g+1}. \tag{2.6}$$

(Konno) [Kon99] The bound (2.6) holds for any fibration of odd genus whose general fibre has general gonality.

Influence of the relative irregularity

Another interesting problem is to study the influence on the slope of *global* invariants, such as the irregularity q of the surface X. It seems that in this context the natural variable to consider is $q - b$, which we will call *relative irregularity.* We have the following results

Xiao ([Xia87a]) If $f : X \to B$ is a non-Albanese fibration, then

$$s(f) \geq 4.$$

Moreover, if $s(f) = 4$ then $q = b + 1$.

Konno in [Kon94] proves the following bound

$$s(f) \geq \frac{4g(g-1)}{(2g-1)(g-q+b)},$$

which is an increasing function of $q - b$. However, notice that this bound is infinitesimal for $g >> 0$.

In [BZ01] Barja and Zucconi obtain a (rather complicated) bound forr non-double cover fibrations, which is an increasing function of $q - b$ (see Theorem 0.7), and which is greater than 4 for $q - b \geq 2$.

2.1.3 Results of the chapter

In section 2.2 we give a new proof of the slope inequality (2.1), using the method of Cornalba and Harris in the generalisation given in the first chapter.

In section 2.3 we give an affirmative answer to Conjecture 2.1.4 on the slope of double cover fibrations. Our proof make an essential use of the Algebraic Index Theorem.

In the last section we approach the problem of studying the influence of the relative irregularity on the slope. We prove with Cornalba-Harris Theorem that $s(f) \geq 4$ if $q - b \geq 2$, and we make some remarks on possible improvements of this result.

2.2 The slope inequality

As observed in section 2.1, Cornalba and Harris in [CH88] proved the slope inequality using their theorem only for non-hyperelliptic semi-stable fibrations. Using the generalised version of their theorem, we are able to prove the slope inequality for any relatively minimal fibration. For non-hyperelliptic fibrations, the proof is almost identical to the original one of Cornalba and Harris, except that we have to prove a vanishing result for the higher direct image sheaf $R^1 f_* \omega_f^h$ (for large enough h). In order to do this, we prove a more general vanishing result for the higher direct image of line bundles on fibred surfaces (Proposition 2.2.1). In the case of hyperelliptic fibrations, the generalisation that we made of the theorem is essential. The main problem is to prove the semi-stability condition, and to handle the sheaf $\mathcal{G}_h = im(\mathrm{Sym}^h f_*\omega_f \to f_*\omega_f^h)$. As for the first problem, we see that the semi-stability of the general fibres reduces to the semi-stability of the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$, and then apply a result of Kempf. On the other hand, to deal with $\mathcal{G}_h$, we manage to reduce ourselves to a double cover of a genus 0 fibration (with a standard construction that will be generalised in section 2.3), and to interpret the sheaf $\mathcal{G}_h$ using the natural subsheaves of $f_*\omega_f$ arising from this construction.

2.2.1 Slope for non-hyperelliptic fibrations

Here we apply Theorem 1.5.1, or more precisely Corollary 1.5.4 to a non-hyperelliptic relatively minimal fibration choosing $L = \omega_f$ and $\mathcal{F} = f_*L$. The conditions of the theorem are satisfied for infinitely many positive values of h, simply because the general fibre X_b is smooth, and the restriction of ω_f to X_b is the canonical sheaf ω_{X_b}. Indeed, we have shown in section 1.3, Corollary 1.3.5, that for a smooth non-hyperelliptic curve, the canonical embedding is Hilbert stable.

Now we want to compute inequality (1.13). First we need to understand the contribution of the higher direct image of ω_f. which is 0 if and only if $\deg R^1 f_*\omega_f^h$ is $O(h)$; in fact, we shall show that $R^1 f_*\omega_f^h$ vanishes applying the following result (whose proof is given below).

Proposition 2.2.1. *Let L be a line bundle on a relatively minimal fibration $f\colon X \to B$. Suppose that*

1) the relative degree of L is strictly positive;

2) if C is an irreducible component of a fibre, then $\deg L_{|C} \geq 0$; moreover, $\deg L_{|C} > 0$ unless C is a (-2)-curve.

Then $R^1 f_* L^h = 0$ for $h >> 0$.

As is well known, $\deg \omega_{f|X_b} = 2g - 2 > 0$, and if D is an irreducible component of a fibre, $\deg \omega_{f|D} \geq 0$, equality holding if and only if D is a (-2)-curve. Therefore the relative canonical sheaf ω_f satisfies the assumptions of the proposition.

Remark 2.2.2. It should be noticed that the vanishing of the direct image sheaf $R^1 f_* \omega_f^h$ can alternatively be derived from the relative version of the Kawamata-Viehweg Theorem proved in [KMM87] (Theorem 1.2.3).

We are now ready to apply Theorem 1.5.1. Observe that $\mathrm{rank} f_* \omega_f = h^0(\omega_{f|X_b}) = g$, where b is general. Inequality (1.13) becomes

$$g(\omega_f \cdot \omega_f) \geq 4(g-1) \deg f_* \omega_f,$$

which is exactly the slope inequality.

Proof of Proposition 2.2.1

While this result is trivial when all fibres are reduced, when dealing with a general fibration one has to be careful in handling non-reduced fibres, as we will see.
By the theory of base change (cf. [Har77], III.2) it is sufficient to prove that

$$h^1(X_b, L^h_{|X_b}) = 0$$

for large enough h for every $b \in B$. Clearly, this is true on general (smooth) fibres, so $R^1 f_* L^h$ is at most a torsion sheaf. By duality for embedded curves (cf. [BHPdV04], II.6)

$$h^1(X_b, L^h_{|X_b}) = h^0(X_b, (\omega_f \otimes L^{-h})_{|X_b}).$$

We therefore need to prove that this last number is 0 for *every* $b \in B$. First of all we need to recall the following result about fibrations.

Lemma 2.2.3. (Zariski's Lemma) Let F be a fibre of a fibration. Let $\{C_i\}_{i \in I}$ be the set of all its irreducible components. Then we have:
1) $C_i F = 0$ for all $i \in I$.
2) If $D = \sum m_i C_i$, $m_i \in \mathbb{Z}$, then $(D)^{\cdot 2} \leq 0$.
3) If D is as above, $(D)^{\cdot 2} = 0$ if and only if $D = rF$ for some $r \in \mathbb{Q}$.

For the proof see for instance [BHPdV04], Lemma III(8.2).

Definition 2.2.4. *A compact connected divisor D on a smooth surface is 1-connected if*

$$(C_1 \cdot C_2) \geq 1$$

for any effective decomposition $D = C_1 + C_2$.

An immediate application of Zariski's Lemma is the following

Lemma 2.2.5. *Let F be a fibre of a fibration. Then $F = mE$, where $m \in \mathbb{Z}$, $m > 0$, and E is 1-connected.*

Proof. Write F as mE, where m is the greatest common divisor of the multiplicities of the fibres of F. We are going to prove that E is 1-connected. Suppose there is an effective decomposition $E = C_1 + C_2$ such that $(C_1 \cdot C_2) \leq 0$. By point (1) of Zariski's Lemma $(C_1)^{\cdot 2} = -(C_1 \cdot C_2) \geq 0$; by (2) $(C_1)^{\cdot 2}$ has to be 0, and using (3) we deduce that there exist $p, q \in \mathbb{Z}$, $q \neq 0$, such that $qC_1 = pF = pmE$. As C_1 is strictly contained in E, $q > pm$. On the other hand q has to divide pm, and we get a contradiction. □

With all this settled, Proposition 2.2.1 is easily implied by the following result:

Lemma 2.2.6. *Let F be a fibre of a fibration and $D = rF$, $r \in \mathbb{Z}$, $r > 0$. Let $\mathcal{L}$ be a line bundle on D with negative total degree and non-positive degree on every irreducible component of D. Then $H^0(D, \mathcal{L}) = 0$.*

Proof. according to Lemma 2.2.5 D has to be of the form mE with E 1-connected, $m > 0$. We split the proof in two parts:

A) $m = 1$: so D is 1-connected. Let s be a section of $H^0(D, \mathcal{L})$. Suppose that $s \neq 0$. Choose a decomposition $D = C_1 + C_2$ such that $C_1 \leq D$ is maximal with respect to the property $s_{|C_1} \equiv 0$. This is an effective decomposition, unless s is nowhere zero. Consider the map of sheaves $\mathcal{O}_D \to \mathcal{L}(-C_1)$ associated to the section s and tensor it with $\mathcal{O}_{C_2}$:

$$\mathcal{O}_{C_2} \to \mathcal{L} \otimes \mathcal{O}_{C_2}(-C_1).$$

By the maximality property of C_1, this morphism is injective. Let $\mathcal{Q}$ be its cokernel. Form the exact sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L} \to \mathcal{Q} \to 0. \tag{2.7}$$

As the first two sheaves are locally free of rank 1 over C_2, the sheaf $\mathcal{Q}$ is torsion. Therefore $\deg \mathcal{Q} = h^0(\mathcal{Q}) \geq 0$. If s were nowhere zero, C_1 would coincide with D and $\mathcal{Q}$ would be 0. This would imply that $\mathcal{L} \cong \mathcal{O}_D$ and in particular that $\deg \mathcal{L} = 0$, a contradiction. Suppose that s vanishes somewhere. Using the Riemann-Roch theorem, we obtain that

$$-(C_1 \cdot C_2) = \deg \mathcal{O}_{C_1}(-C_2) > \deg \mathcal{O}_{C_1}(-C_2) + \deg \mathcal{L}_{|C_2} =$$

$$= \chi(\mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L}) - \chi(\mathcal{O}_{C_2}) = h^0(\mathcal{Q}) \geq 0,$$

contrary to the assumption of 1-connectedness. We have thus proved that $H^0(D, \mathcal{L}) = 0$ in the 1-connected case (notice that in this part of the proof we haven't used the fact that F is a fibre of a fibration; the statement holds for *any* 1-connected D).

B) $m > 1$: we proceed by induction on m. Fix i such that $1 < i \leq m$, and suppose that $H^0((i-1)E, \mathcal{L}) = 0$. Consider the decomposition sequence (cf. [BHPdV04] pag.62)

$$0 \to \mathcal{O}_E(-(i-1)E) \to \mathcal{O}_{iE} \to \mathcal{O}_{(i-1)E} \to 0$$

and tensor it with $\mathcal{L}$.

$$0 \to \mathcal{L} \otimes \mathcal{O}_E(-(i-1)E) \to \mathcal{L} \otimes \mathcal{O}_{iE} \to \mathcal{L} \otimes \mathcal{O}_{(i-1)E} \to 0$$

Passing to cohomology we get

$$0 \to H^0(E, \mathcal{L} \otimes \mathcal{O}_E(-(i-1)E)) \to H^0(iE, \mathcal{L}) \to H^0((i-1)E, \mathcal{L}) \to ...$$

The last space is 0 by the induction hypothesis. As for the first one, observe that the line bundle $\mathcal{L}_{|E} \otimes \mathcal{O}_E(-(i-1)E)$ satisfies the condition of part (A) because E is 1-connected,

$$\deg \mathcal{L}_{|E} \otimes \mathcal{O}_E(-(i-1)E) = \deg \mathcal{L}_E + \deg \mathcal{O}_E(-(i-1)E) = \deg \mathcal{L}_E = \frac{1}{m} \deg \mathcal{L}_D < 0$$

and its degree on the connected components of E equals the degree of $\mathcal{L}$ by Zariski's Lemma. Applying part (A) concludes the proof. □

It is clear that $(\omega_f \otimes L^{-h})_{|X_b}$ satisfies the conditions of Lemma 2.2.6 when h is large enough. Proposition 2.2.1 is thus proved.

2.2.2 Slope for hyperelliptic fibrations

Let $f : X \to B$ be a hyperelliptic fibration and X_b a general fibre. The canonical line bundle $\omega_{X_b} = \omega_{f|X_b}$ induces a morphism ψ to $\mathbb{P}^{g-1}$ that factors as follows:

$$X_b \xrightarrow{\varphi} \mathbb{P}^1 \xrightarrow{v} \mathbb{P}^{g-1}$$

where φ is a double cover ramified at the Weierstrass points of X_b and v is the Veronese embedding of degree $g-1$. Note that

$$\omega_{X_b} = \psi^*(\mathcal{O}_{\mathbb{P}^{g-1}}(1)) = \varphi^*(\mathcal{O}_{\mathbb{P}^1}(g-1)),$$

and that $\mathrm{Sym}^h H^0(X_b, \omega_{X_b})$ can be identified with $H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(h))$. Moreover, the fibre of the morphism

$$\mathrm{Sym}^h f_*\omega_f \xrightarrow{\gamma_h} \mathcal{G}_h \hookrightarrow f_*\omega_f^h$$

at b can be identified with

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(h)) \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h(g-1))) \hookrightarrow H^0(X_b, \omega_{X_b}^h). \tag{2.8}$$

In particular, rank $\mathcal{G}_h = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h(g-1))) = h(g-1)+1$.

Double cover construction

A general hyperelliptic fibred surface is not always a double cover of a genus 0 fibration. Anyway, we show below that for our purposes it can be treated as if it were . The following is a simplified version of the construction for double cover fibrations given in section 2.11.

Since the fibration is relatively minimal and the relative canonical map is a generically finite rational map of degree two, X has an involution ι which restricts to the hyperelliptic involution on general fibres. If ι has no isolated fixed points then $X/\langle\iota\rangle$ is a smooth ruled surface on B and the quotient map is a double cover whose ramification divisor is the fixed locus of ι. Otherwise, we blow up the isolated fixed points and obtain a smooth surface $\widetilde{X}$ birational to X whose induced involution $\tilde{\iota}$ has no isolated fixed points. Call W the quotient of $\widetilde{X}$ by $\tilde{\imath}$. The surface W has a natural ruling over B, but is not relatively minimal. We have the following diagram:

$$\begin{array}{ccc} \widetilde{X} & & \\ \eta \downarrow & \searrow^{\alpha} & \\ X & \xrightarrow{\psi} & W \\ f \downarrow & \swarrow_{\beta} & \\ B & & \end{array}$$

Let $R \subset W$ be the branch divisor of α. By the theory of cyclic coverings we can find a line bundle $\mathcal{L}$ on W such that $\mathcal{L}^2 = \mathcal{O}_W(R)$.

Set $\tilde{f} = f \circ \eta$. Recall that $\omega_{\tilde{f}} = \eta^*\omega_f \otimes \mathcal{O}_{\widetilde{X}}(E)$, where E is the (disjoint) union of the exceptional (-1)-curves. Let ϵ be the number of components of E (i.e. the number of blow ups of which η is made of). Consider the exact sequence

$$0 \to \eta^*\omega_f^h \to \omega_{\tilde{f}}^h \to \mathcal{O}_{hE}(hE) \to 0$$

and the long exact sequence induced by the pushforward by $\tilde{f}$:

$$0 \to f_*\omega_f^h \to \tilde{f}_*\omega_{\tilde{f}}^h \to \tilde{f}_*\mathcal{O}_{hE}(hE) \to ...$$

$$... \to R^1f_*\omega_f^h \to R^1\tilde{f}_*\omega_{\tilde{f}}^h \to R^1\tilde{f}_*\mathcal{O}_{hE}(hE) \to 0.$$

Observe that $\deg \tilde{f}_*\mathcal{O}_{hE}(hE) = h^0(\mathcal{O}_{hE}(hE)) = 0$, and that

$$\deg R^1\tilde{f}_*\mathcal{O}_{hE}(hE) = h^1(\mathcal{O}_{hE}(hE)) = \epsilon\,\frac{h^2+h}{2},$$

by the Riemann-Roch theorem for embedded curves. Therefore $\tilde{f}_*\omega_{\tilde{f}}^h = f_*\omega_f^h$, and

$$\deg R^1\tilde{f}_*\omega_{\tilde{f}}^h = \deg R^1f_*\omega_f^h + \epsilon\,\frac{h^2+h}{2} = \epsilon\,\frac{h^2+h}{2}.$$

Recall that in our situation $\omega_{\tilde{f}} = \alpha^*(\omega_\beta \otimes \mathcal{L})$ (see [BHPVdV] Chap.I, Lemma (17.1)), and $\alpha_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_W \oplus \mathcal{L}^{-1}$ (see [BHPVdV] Chap.I, Lemma (17.12)). We therefore have the following chain of isomorphisms

$$f_*\omega_f = \tilde{f}_*\omega_{\tilde{f}} = \beta_*\alpha_*\omega_{\tilde{f}} = \beta_*\alpha_*(\alpha^*(\omega_\beta \otimes \mathcal{L})) = \beta_*\left((\omega_\beta \otimes \mathcal{L}) \otimes \alpha_*\mathcal{O}_{\widetilde{X}}\right) =$$

$$= \beta_*\left((\omega_\beta \otimes \mathcal{L}) \otimes (\mathcal{O}_W \oplus \mathcal{L}^{-1})\right) = \beta_*\left((\omega_\beta \otimes \mathcal{L}) \oplus \omega_\beta\right) = \beta_*(\omega_\beta \otimes \mathcal{L});$$

and the inclusion:

$$\beta_*(\omega_\beta \otimes \mathcal{L})^h \hookrightarrow \beta_*(\omega_\beta \otimes \mathcal{L})^h \oplus \beta_*(\omega_\beta^h \otimes \mathcal{L}^{h-1}) = \beta_*\left((\omega_\beta \otimes \mathcal{L})^h \otimes (\mathcal{O}_W \oplus \mathcal{L}^{-1})\right) = \tilde{f}_*\omega_{\tilde{f}}^h = f_*\omega_f^h.$$

Form the following diagram of sheaves on B

$$\begin{array}{ccccc} \mathrm{Sym}^h\beta_*(\omega_\beta \otimes \mathcal{L}) & \xrightarrow{\sim} & \mathrm{Sym}^h\tilde{f}_*\omega_{\tilde{f}} & \xrightarrow{\sim} & \mathrm{Sym}^h f_*\omega_f \\ \downarrow & & \downarrow & & \downarrow \\ \beta_*(\omega_\beta \otimes \mathcal{L})^h & \hookrightarrow & \tilde{f}_*\omega_{\tilde{f}}^h & \xrightarrow{\sim} & f_*\omega_f^h \end{array}$$

For general $b \in B$ the fibre $\beta_*(\omega_\beta \otimes \mathcal{L})^h \otimes \boldsymbol{k}(b)$ is $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h(g-1)))$; we hence *choose* $\beta_*(\omega_\beta \otimes \mathcal{L})^h$ *as the sheaf* $\mathcal{G}_h$ in Crollary 1.5.4 of Theorem 1.5.1.

In the following we prove that the conditions of the theorem hold and then perform the computation of inequality (1.12), which becomes exactly the slope inequality.

Semi-stability in the hyperelliptic case

Let X_b be a general fibre of the fibration. We want to show that the homomorphism

$$\mathrm{Sym}^h H^0(\omega_{f|X_b}) \twoheadrightarrow \mathcal{G}_h \otimes \boldsymbol{k}(b)$$

is semi-stable for large enough h. As observed in the first chapter, and can be easily checked using diagram (2.8), this homomorphism coincides with the one induced by the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$:

$$\mathrm{Sym}^h H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1)) \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h(g-1))).$$

The $SL(g, \mathbb{C})$-action is induced by the isomorphism $H^0(X_b, \omega_{X_b}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1))$. We are therefore reduced to verifying the semi-stability of this embedding. This can be derived from the following powerful result (cf. [Kem78], corollary 5.3):

Theorem 2.2.7. (Kempf) *Complete homogeneous spaces under the action of linear algebraic groups of characteristic 0 embedded by complete linear systems have semi-stable Hilbert points.*

Remark 2.2.8. The Veronese embedding in $\mathbb{P}^{g-1}$ is linearly semi-stable but *not* linearly stable (cf. [Mum77]). Therefore, we cannot use Theorem 1.4.2, nor Mumford's results mentioned in Remark 1.4.6.

Conclusion: the slope inequality in the hyperelliptic case

Observe that

$$(\omega_f \cdot \omega_f) - \epsilon = (\omega_{\tilde{f}} \cdot \omega_{\tilde{f}}) = (\alpha^*(\omega_\beta \otimes \mathcal{L}) \cdot \alpha^*(\omega_\beta \otimes \mathcal{L})) = 2(\omega_\beta \otimes \mathcal{L} \cdot \omega_\beta \otimes \mathcal{L}),$$

because α is a finite morphism of degree 2. Using Grothendieck-Riemann-Roch we compute the degree of $\mathcal{G}_h$:

$$\deg \mathcal{G}_h = \frac{h^2}{2}(\omega_\beta \otimes \mathcal{L} \cdot \omega_\beta \otimes \mathcal{L}) + \deg R^1\beta_*(\omega_\beta \otimes \mathcal{L})^h + O(h).$$

We now estimate the degree of $R^1\beta_*(\omega_\beta \otimes \mathcal{L})^h$ for $h >> 0$. Observe that $R^1\tilde{f}_*\omega_{\tilde{f}}^h$ is torsion and splits into the direct sum

$$R^1\tilde{f}_*\omega_{\tilde{f}}^h = R^1\beta_*(\omega_\beta \otimes \mathcal{L})^h \oplus R^1\beta_*(\omega_\beta^h \otimes \mathcal{L}^{h-1}).$$

Hence $\deg R^1\beta_*(\omega_\beta \otimes \mathcal{L})^h \leq \deg R^1\tilde{f}_*\omega_{\tilde{f}}^h = \epsilon\frac{h^2+h}{2}$ Putting everything together inequality (1.12) becomes

$$0 \leq \frac{g}{2}\left((\omega_\beta \otimes \mathcal{L} \cdot \omega_\beta \otimes \mathcal{L}) + \epsilon\right) - (g-1)\deg \tilde{f}_*\omega_{\tilde{f}} \leq \frac{g}{4}(\omega_f \cdot \omega_f) - (g-1)\deg f_*\omega_f$$

and the slope inequality is proved in the hyperelliptic case as well.

2.3 The slope of double fibrations

In this section, we give an affirmative answer to Barja's conjecture 2.1.4 on the slope of double fibrations (Theorem 2.3.9 and Theorem 2.3.10), together with a characterisation of the fibrations that reach the bound. Our results follow from an application of the slope inequality for fibred surfaces and of the Algebraic Index Theorem, or Signature Theorem (see e.g. [BHPdV04] Theorem IV.2.14 or [GH78]).

In 2.3.1 we discuss the problem of gluing involutions on the general fibres to a global involution. In 2.3.2 we apply the method of Cornalba and Harris to this case, finding that we can extend very naturally the argument used in section 2.2 for hyperelliptic fibrations. We obtain with this strategy an inequality on the invariants of the fibration, already found by Barja in [Bar] with Xiao's method, which however is not sharp. In 2.3.3 we recall a construction that allows to relate the invariants of a double cover fibration to the ones of a fibration which is a "true" double cover of a relatively minimal fibration of genus γ. The rest of the section is devoted to the proof of the bound and to the characterisation of the extremal case, while in 2.3.5 we present two examples that prove the sharpness of the bound.

2.3.1 Double covers and double fibrations

Recall that a double cover of surfaces is a finite morphism $V \to W$ of degree two between surfaces. It is determined by its branch divisor $R \subset W$, and there is a line bundle $\mathcal{L}$ on W such that $\mathcal{L}^2 = \mathcal{O}_W(R)$. Suppose that W is smooth; then V is normal (resp., smooth) if and only if R is reduced (resp., smooth). For the theory of double covers we refer to [BHPdV04] and [Mat90].

We give the definition of double fibrations in analogy with the definition of hyperelliptic and bi-elliptic ones.

Definition 2.3.1. *A* double fibration of type (g, γ) *is a relatively minimal genus g fibred surface $f\colon X \to B$ such that there is a degree 2 morphism from the general fibre of f to a smooth curve of genus γ.*

A double fibration need not be a double cover of a fibration, not even up to birational isomorphism. The main problem is that it is possible that the involutions on the general fibres don't glue together to give a global involution on X. This obstacle can be overcome with a base change; indeed, after a possibly ramified base change $T \to B$, one can always find a fibred surface $\alpha\colon Y \to T$ and a birational model of $X \times_B T$ which is a double cover of Y (see [BN98]). The drawback in this approach is that, according to Tan's results (see Theorem 2.1.1), the control of the slope is lost with a base change that ramifies over fibres which are not moduli stable. Indeed, Theorem 2.1.1 has for example the following consequence. Suppose we are given a fibration with special fibres that are multiples of nodal curves, and we perform semi-stable reduction and obtain a semi-stable model $\overline{f} : \overline{X} \to \overline{B}$. Suppose in addition that we can prove that $s(\overline{f}) \geq 8$. Then we know that also $s(f) \geq 8$. But in the case of double fibrations, all we get is $s(\overline{f}) \geq 4(g-1)/(g-\gamma)$, and the last number is strictly smaller than 8, so we can not conclude anything on the slope of the original fibration.

It is therefore more appropriate, when dealing with the general case, to use the following definition:

Definition 2.3.2. *A* double cover fibration of type (g, γ) *is the datum of a genus g fibration $f : X \to B$ together with a global involution on X that restricts, on the general fibre, to an involution with genus γ quotient.*

Remark 2.3.3. In [BZ01] the authors use the term "double cover fibration" to denote a slightly more general concept: a double fibration with a rational degree 2 morphism to a relatively minimal fibred surface. We will call this a *birational double cover fibration.* As we see below, in the case $g > 4\gamma + 1$ these definitions coincide. The bound $4(g-1)/(g-\gamma)$ can be proven also with this definition, as we see at the end of section 2.3.3 (cf. [BZ01])

However, given a double fibration, if the involution on the general fibre is unique, arguing again as in [BN98], we can get a global involution on X as above; hence, in this case, there is no difference between Definition 2.3.1 and Definition 2.3.2.

We shall show below that, under the assumption $g > 4\gamma+1$, the involution of F is indeed unique, and that the same happens when $g = 4\gamma + 1$, except in a very special case. The argument is due to Barja (Lemma 4.7 in [Bar]), save for the discussion of the case $g = 4\gamma+1$.

Lemma 2.3.4. *Let F be a smooth curve of genus g, and let $\gamma \geq 1$ be an integer. If $g > 4\gamma+1$, then F has at most one involution ι such that $\Gamma = F/\langle\iota\rangle$ has genus $\leq \gamma$. If instead $g = 4\gamma+1$, and there are distinct involutions ι_1 and ι_2 of F such that the quotients $F/\langle\iota_1\rangle = \Gamma_1$, $F/\langle\iota_2\rangle = \Gamma_2$ have genera γ_1, γ_2 both not exceeding γ, then $\gamma_1 = \gamma_2 = \gamma$, Γ_1 and Γ_2 are hyperelliptic, the natural map $F \to \Gamma_1 \times \Gamma_2$ is an embedding, and its image belongs to the linear system $|\pi_1^*(2q_1) + \pi_2^*(2q_2)|$, where q_i is a Weierstrass point on Γ_i and π_i denotes the projection $\Gamma_1 \times \Gamma_2 \to \Gamma_i$.*

Proof. Suppose ι_1 and ι_2 are two involutions of F such that the quotients

$$F/\langle\iota_1\rangle = \Gamma_1, \quad F/\langle\iota_2\rangle = \Gamma_2$$

have genera γ_1 and γ_2 not greater than γ. Consider the commutative diagram

$$\begin{array}{ccccc} & & F & & \\ & \sigma_1 \swarrow & \downarrow \sigma & \searrow \sigma_2 & \\ \Gamma_1 & \xleftarrow{\beta_1} & D & \xrightarrow{\beta_2} & \Gamma_2 \\ & \pi_1 \nwarrow & \cap & \nearrow \pi_2 & \\ & & \Gamma_1 \times \Gamma_2 & & \end{array}$$

where the σ_i are the quotient morphisms, the π_i are the projections, $\sigma = \sigma_1 \times \sigma_2$ and $D = \sigma(F)$. Clearly, the degree of σ is either 1 or 2. If it is 2, the β_i have to be isomorphisms; therefore σ_1 and σ_2 are the quotient maps of the *same* involution on F. Conversely, if the involutions ι_1 and ι_2 coincide, the degree of σ must be 2.

Now suppose that $\deg \sigma = 1$. Set $L_i = \pi_i^{-1}(p_i) \subseteq \Gamma_1 \times \Gamma_2$, with $p_i \in \Gamma_i$. The effective divisor $L_1 + L_2$ has self-intersection $2 > 0$. By the Index Theorem the determinant of the intersection matrix of the pair $(D, L_1 + L_2)$ has to be non-positive:

$$2(D \cdot D) - (D \cdot L_1 + L_2)^2 \leq 0.$$

Since $(D \cdot L_i) = 2$, we obtain that $(D \cdot D) \leq 8$. By the adjunction formula:

$$2g-2 \leq 2p_a(D)-2 = (K_{\Gamma_1\times\Gamma_2}+D\cdot D) = (\pi_1^*K_{\Gamma_1}+\pi_2^*K_{\Gamma_2}+D\cdot D) \leq 4(\gamma_1+\gamma_2) \leq 8\gamma. \quad (2.9)$$

Thus $g \leq 4\gamma + 1$, and the first part of the lemma is proven.

If $g = 4\gamma + 1$, all the above inequalities must necessarily be equalities. In particular, $\gamma_1 = \gamma_2 = \gamma$, and $p_a(D) = g$, so the morphism σ is an embedding. Furthermore, by the Index Theorem, D has to be numerically (homologically) equivalent to a multiple of $L_1 + L_2$. Intersecting with L_1 and L_2, one sees that $D \equiv 2L_1 + 2L_2$. Hence D is linearly equivalent to a divisor of the form $\pi_1^*A_1 + \pi_2^*A_2$, where A_i is a divisor of degree 2 on Γ_i. Observe that

$$H^0(\Gamma_1 \times \Gamma_2\,, \pi_1^*A_1 + \pi_2^*A_2) = H^0(\Gamma_1, A_1) \otimes H^0(\Gamma_2, A_2).$$

Since D is smooth, the dimension of $H^0(\Gamma_i, A_i)$ must be at least 2. We therefore conclude that the Γ_i are hyperelliptic and that $A_i \sim 2q_i$, where q_i is a Weierstrass point. □

Lemma 2.3.4 implies in particular that, if $g = 4\gamma + 1$ and F has an involution ι such that $F/\langle\iota\rangle$ has genus γ, the involution is unique if $F/\langle\iota\rangle$ is non-hyperelliptic.

If the general fibres of a double fibration are as described in the second part of Lemma 2.3.4, it is possible that a non-trivial base change is needed in order to get a global involution, as the following example shows (see also [Bar01] for an example in the bi-elliptic case).

Example 2.3.5. Let F be a smooth hyperelliptic curve of genus γ. Let B be a smooth curve of positive genus, and let $T \to B$ be an unramified degree two covering; thus B is the quotient of T modulo a base-point-free involution σ. We set $G = \langle\sigma\rangle$, and let σ act on the product $F\times F$ by exchanging the components. Call π_1, π_2 the projections of $F\times F$ to the two factors. Let q be a Weierstrass point of F, and consider the linear system $|\pi_1^*(2q) + \pi_2^*(2q)|^G$ of effective G-invariant divisors linearly equivalent to $\pi_1^*(2q)+\pi_2^*(2q)$; it is immediate to show that it is base-point-free and not composed with an involution. So, by Bertini's theorem, a general member $D \in |\pi_1^*(2q) + \pi_2^*(2q)|^G$ is smooth and irreducible.

Let G act on $F \times F \times T$ by

$$\sigma(f_1, f_2, t) = (f_2, f_1, \sigma(t)).$$

The subvariety $D \times T \subseteq F \times F \times T$ is clearly G-invariant. Set $W = (D \times T)/G$; then $W \to B$ is a double fibration with all fibres isomorphic to D (so it is locally trivial) and genus $g = g(D) = 4\gamma + 1$, but is not a double cover fibration.

2.3.2 Application of the Cornalba-Harris Theorem to double cover fibrations

We can apply Corollary 1.5.4 to the case of double cover fibrations, arguing almost in the same way as we did for the hyperelliptic case in section 2.2, and obtaining the following result

Proposition 2.3.6. *Let $f\colon X \longrightarrow B$ be any double cover fibration of type (g, γ) with $g \geq 2\gamma + 1$. Let $\alpha\colon Y \longrightarrow B$ be the associated fibration of genus γ. Then*

$$s(f) \geq 4\frac{g-1}{g-\gamma}\left(1 - \frac{\deg \alpha_*\omega_\alpha}{\deg f_*\omega_f}\right). \quad (2.10)$$

In particular any double cover over a locally trivial fibration satisfies the expected bound.

Proof. We blow up the isolated fixed points of the involution on X, obtaining a smooth surface $\widetilde{X} \longrightarrow X$ with a double cover π over a smooth surface Y. Let $\tilde{f}\colon \widetilde{X} \longrightarrow B$ be the resulting fibration. The surface Y has a fibration of genus γ,

$$\alpha\colon Y \longrightarrow B.$$

Note that in general α is not relatively minimal; for instance, a (-2)-curve in $\widetilde{X}$ becomes a (-1)-curve in Y. As we verified in 2.2.2, we have

$$\tilde{f}_*\omega_{\tilde{f}}^h = f_*\omega_f^h,$$

for any $h \geq 1$, and

$$(\omega_{\tilde{f}} \cdot \omega_{\tilde{f}}) = (\omega_f \cdot \omega_f) - \epsilon,$$

where ϵ is the number of blow ups from X to $\widetilde{X}$. Moreover,

$$\deg R^1\tilde{f}_*\omega_{\tilde{f}}^h = \deg R^1 f_*\omega_f^h + \epsilon\,\frac{h^2+h}{2} = \epsilon\,\frac{h^2+h}{2}.$$

for large h. Let $R \subset Y$ be the branch divisor of π. By the theory of cyclic coverings, there exists a line bundle $\mathcal{L}$ on Y such that $\mathcal{L}^2 = \mathcal{O}_Y(R)$. Recall that

$$\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$$

and

$$\omega_{\tilde{f}} = \pi^*(\omega_\alpha \otimes \mathcal{L}).$$

Therefore we have the following decomposition of $\tilde{f}_*\omega_{\tilde{f}}$

$$\tilde{f}_*\omega_{\tilde{f}} = \alpha_*\pi_*\omega_f = \alpha_*(\pi_*\pi^*(\omega_\alpha \otimes \mathcal{L})) = \alpha_*((\omega_\alpha \otimes \mathcal{L}) \otimes \pi_*\mathcal{O}_Y) = \alpha_*(\omega_\alpha \otimes \mathcal{L}) \oplus \alpha_*\omega_\alpha,$$

which on the general fibre F amounts to

$$H^0(F, \omega_F) = H^0(\Gamma, \omega_\Gamma(L)) \oplus H^0(\Gamma, \omega_\Gamma).$$

where L is the restriction of $\mathcal{L}$ to Γ. By Hurwitz' formula $\deg L = g - 2\gamma + 1$. We are now going to apply Corollary 1.5.4 of Theorem 1.5.1 to the natural subsheaf $\alpha_*(\omega_\alpha \otimes \mathcal{L})$ of $f_*\omega_f$. In this case

$$\mathrm{rank}\mathcal{F} = h^0(\Gamma, \omega_\Gamma(L)) = g - \gamma.$$

We split the proof in two cases.

1. Suppose that the restriction of $\mathcal{F}$ on a general fibre Γ does not belong to a g^1_2 on Γ (this holds in particular if α is non-hyperelliptic or if $g \geq 2\gamma + 2$). In this case $\mathcal{F}$ induces on a general fibre X_b a $2:1$ morphism to Γ followed by the morphism ψ in $\mathbb{P}^{g-\gamma-1}$ induced by the line bundle $\omega_\Gamma(L)$. We distinguish again two cases.
   (a) ψ is an embedding; in this case it is linearly stable, by [Mum77], section 2.15, hence, by the same argument made the proof of the slope inequality in the non-hyperelliptic case, it is Hilbert stable. We apply Corollary 1.5.4 taking as $\mathcal{G}_h$ the

sheaf $\alpha_*(\omega_\alpha^h \otimes \mathcal{L}^h)$. Now, computing $\deg \mathcal{G}_h$, rank$\mathcal{G}_h$, and $\deg R^1\alpha_*(\omega_\alpha \otimes \mathcal{L})^h$ for $h \gg 0$, inequality (1.13) becomes

$$\frac{g-\gamma}{2}\left(((\omega_\alpha \otimes \mathcal{L} \cdot \omega_\alpha \otimes \mathcal{L}) + \frac{\epsilon}{2}\right) - (g-1)\deg \alpha_*(\omega_\alpha \otimes \mathcal{L}) \geq 0.$$

Remembering that

$$2(\omega_\alpha \otimes \mathcal{L} \cdot \omega_\alpha \otimes \mathcal{L}) = (\omega_{\tilde{f}} \cdot \omega_{\tilde{f}}) = (\omega_f \cdot \omega_f) - \epsilon,$$

and that

$$\deg \alpha_*(\omega_\alpha \otimes \mathcal{L}) = \deg \tilde{f}_*\omega_{\tilde{f}} - \deg \alpha_*\omega_\alpha = \deg f_*\omega_f - \deg \alpha_*\omega_\alpha,$$

we obtain the statement.

(b) ψ fails to be an embedding if and only if $\deg L = 2$. Note that, by assumption, if C is hyperelliptic, $L \notin g^1_2$. In this case ψ is a birational morphism, which is linearly semistable, and hence, by [Mum77] again, its image is Chow semistable. Chow semistability does not imply Hilbert semistability, hence we cannot use the Cornalba-Harris method; however, we can in this case apply a result of Bost ([Bos94], Theorem 3.3) that gives as a consequence exactly the same inequality of Corollary 1.5.4.

2. Suppose on the other hand that α is hyperelliptic and that the morphism induced by $\alpha_*\omega_\alpha \otimes \mathcal{L}$ on a general fibre factors through the hyperelliptic involution of Γ:

$$X_b \xrightarrow{2:1} \Gamma \xrightarrow{2:1} \mathbb{P}^1 \overset{v}{\hookrightarrow} \mathbb{P}^{g-\gamma-1},$$

where v is the Veronese embedding. The semistability assumption is satisfied because v is Hilbert semistable (as observed in the hyperelliptic case) With similar computations, we obtain

$$\deg \mathcal{G}_h = h^2\frac{(\omega_f \cdot \omega_f)}{8} + O(h), \quad \text{rank}\mathcal{G}_h = \gamma h + O(1),$$

and again inequality (1.13) gives the desired bound.

□

Remark 2.3.7. This inequality was found by Barja ([Bar], Prop.4.10(iii)), using Xiao's method on the same vector bundle $\alpha_*(\omega_\alpha \otimes \mathcal{L})$.

2.3.3 Reduction to double covers of relatively minimal fibrations

In order to prove the expected bound for general double covers, we need to use the slope inequality on the fibration of genus γ. In other words we need to reduce ourselves to double covers over relatively minimal fibrations. This can be done using the following well-known construction, of which we present a quick overview. The precise construction can be found for instance in [AK00] or in [BZ01].

Let $f\colon X\to B$ be a double cover fibration of type (g,γ). Let ι be the involution on X. If it has a fixed locus of codimension 1, the quotient $X/\langle\iota\rangle$ is a smooth surface. Otherwise consider the blow-up of X at the isolated fixed points of ι:

$$\widetilde{X}\longrightarrow X$$

and call $\tilde{\iota}$ the induced involution on it. The quotient $\widetilde{X}/\langle\tilde{\iota}\rangle=\widetilde{Y}$ is a smooth surface with a natural fibration $\tilde{\alpha}$ over B which is not necessarily relatively minimal. Let $\alpha\colon Y\to B$ be its minimal model.

$$\begin{array}{ccccc}\widetilde{X} & \longrightarrow & \widetilde{Y} & \longrightarrow & Y\\ \downarrow & & \tilde{\alpha}\downarrow & \swarrow\alpha & \\ X & \xrightarrow{f} & B & & \end{array}$$

The direct image R of the branch locus of the double cover $\widetilde{X}\to\widetilde{Y}$ induces a double cover $X'\to Y$, with X' normal but not necessarily smooth; notice however that, by construction, X' is locally a hypersurface in a smooth threefold, so its dualizing sheaf is locally free and the Grothendieck-Riemann-Roch Theorem still holds on it. To obtain a smooth double cover we perform the *canonical resolution* (see [BHPdV04] III.7, [Bar01] sec. 2, [AK00] sec. 2.2)

$$\begin{array}{cccccccccc}X_k & \xrightarrow{\sigma_k} & X_{k-1} & \longrightarrow & \dots & \longrightarrow & X_1 & \xrightarrow{\sigma_1} & X_0 & =X'\\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\ Y_k & \xrightarrow{\tau_k} & Y_{k-1} & \longrightarrow & \dots & \longrightarrow & Y_1 & \xrightarrow{\tau_1} & Y_0 & =Y\end{array}$$

where the τ_j are successive blow-ups that resolve the singularities of R; the morphism $X_j\to Y_j$ is the double cover with branch locus

$$R_j:=\tau_j^*R_{j-1}-2\left[\frac{m_{j-1}}{2}\right]E_j,$$

where E_j is the exceptional divisor of τ_j, m_{j-1} is the multiplicity of the blown-up point, and $[\ \ ]$ stands for integral part. Let $f_j\colon X_j\to B$, $f'\colon X'\to B$ be the induced fibrations. A computation shows that

$$(\omega_{f_k}\cdot\omega_{f_k})=(\omega_{f'}\cdot\omega_{f'})-2\sum_{i=1}^{k}\left(\left[\frac{m_i}{2}\right]-1\right)^2,$$

and that

$$\deg(f_k)_*\omega_{f_k}=\deg f'_*\omega_{f'}-\frac{1}{2}\sum_{i=1}^{k}\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right).$$

Observe that, since X_k is smooth, by the relative minimality of $f\colon X\to B$ there is a morphism $\beta\colon X_k\to X$. Therefore

$$(\omega_f\cdot\omega_f)=(\omega_{f_k}\cdot\omega_{f_k})+\epsilon,$$

where ϵ is the number of blow-ups which make up β. Moreover, observe that $f_*\omega_f = (f_k)_*\omega_{f_k}$. Hence we get the following fundamental identity:

$$\begin{aligned}(\omega_f \cdot \omega_f) - 4\frac{g-1}{g-\gamma}\deg f_*\omega_f = \\ = (\omega_{f'} \cdot \omega_{f'}) - 4\frac{g-1}{g-\gamma}\deg f'_*\omega_{f'} + 2\sum_{i=1}^{k}\left(\left[\frac{m_i}{2}\right]-1\right)\left(\frac{\gamma-1}{g-\gamma}\left[\frac{m_i}{2}\right]+1\right)+\epsilon.\end{aligned} \tag{2.11}$$

Definition 2.3.8. *In the situation above, we say that the branch divisor $R \subset Y$ has* negligeable singularities *if all the multiplicities in the above process equal 2 or 3 (cf. [Per82])*

The birational double cover case

When dealing with a birational double cover fibration (cf. Remark 2.3.3), we can still obtain an inequality of the same type using Stein's Theorem as follows (this argument is due to Barja and Zucconi; see [BZ01]).

Suppose $f : X \to B$ is a birational double cover fibration. By definition there is a rational map of degree two on a relatively minimal fibration of genus γ $\pi : V \to B$. Blowing up X one can get a generically 2-1 morphism $\widetilde{\pi} : \widetilde{X} \to B$ fitting in the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\pi}} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

By Stein's Theorem there exists a normal surface Z, a finite morphism $\pi_0 : Z \to V$ and a birational morphism $u : \widetilde{X} \to Z$ such that the map $\widetilde{\pi}$ factors by as $\pi_0 \circ u$. Now it is possible to apply the canonical resolution to $\pi_0 : Z \to V$ and, arguing as above, obtain equality (2.11).

2.3.4 The bound

Theorem 2.3.9. *Let $f\colon X \to B$ be a double fibration of type (g, γ). If $g > 4\gamma + 1$, then*

$$(\omega_f \cdot \omega_f) \geq 4\frac{g-1}{g-\gamma}\deg f_*\omega_f \tag{2.12}$$

If $\gamma \geq 1$, equality holds if and only if X is the minimal desingularisation of a double cover $\pi\colon \overline{X} \to Y$ of a locally trivial genus γ fibration $\alpha\colon Y \to B$ such that the branch locus R of π has only negligeable singularities and, in addition, when $\gamma > 1$, is numerically equivalent to a linear combination of ω_α and a fibre of α.

Proof. The case $\gamma = 0$ is the slope inequality for hyperelliptic fibrations. The case $\gamma = 1$ has been proved in [Bar01]. We therefore assume that $\gamma > 1$. In view of Lemma 2.3.4, the assumptions about g and γ guarantee that we are in fact dealing with a double cover fibration. The first part of the argument applies to any double cover fibration, even without the assumption that $g > 4\gamma + 1$. We adopt the notation introduced in the previous section.

In view of the identity (2.11), to prove (2.12) it suffices to prove its analogue for $f'\colon X' \to B$. Recall that the double covering $\xi\colon X' \to Y$ corresponds to a line bundle $\mathcal{L}$ on Y such that $\mathcal{L}^2 = \mathcal{O}(R)$, where R is the ramification divisor of ξ, and that

$$\xi_*\mathcal{O}_{X'} = \mathcal{O}_Y \oplus \mathcal{L}^{-1}\,, \quad \omega_{f'} = \xi^*(\omega_\alpha \otimes \mathcal{L})\,.$$

It follows that

$$(\omega_{f'} \cdot \omega_{f'}) = 2(\omega_\alpha \otimes \mathcal{L} \cdot \omega_\alpha \otimes \mathcal{L}) = 2(\omega_\alpha \cdot \omega_\alpha) + 4(\mathcal{L} \cdot \omega_\alpha) + 2(\mathcal{L} \cdot \mathcal{L}),$$

and also, by the Riemann-Roch theorem, that

$$\deg {f'}_*\omega_{f'} = 2 \deg \alpha_*\omega_\alpha + \frac{(\mathcal{L} \cdot \mathcal{L})}{2} + \frac{(\mathcal{L} \cdot \omega_\alpha)}{2}.$$

Hence we may write:

$$(\omega_{f'} \cdot \omega_{f'}) - 4\frac{g-1}{g-\gamma} \deg {f'}_*\omega_{f'} =$$

$$= 2\left((\omega_\alpha \cdot \omega_\alpha) - 4\frac{g-1}{g-\gamma} \deg \alpha_*\omega_\alpha\right) - 2\frac{\gamma-1}{g-\gamma}(\mathcal{L} \cdot \mathcal{L}) + 2\frac{g-2\gamma+1}{g-\gamma}(\mathcal{L} \cdot \omega_\alpha).$$

Using the slope inequality (2.1) for $\alpha\colon Y \to B$ we obtain that

$$(\omega_\alpha \cdot \omega_\alpha) - 4\frac{g-1}{g-\gamma} \deg \alpha_*\omega_\alpha \geq \frac{-g-\gamma^2+2\gamma}{(\gamma-1)(g-\gamma)}(\omega_\alpha \cdot \omega_\alpha).$$

Therefore

$$(\omega_{f'} \cdot \omega_{f'}) - 4\frac{g-1}{g-\gamma} \deg {f'}_*\omega_{f'} \geq$$

$$\geq \frac{2}{g-\gamma}\left((g-2\gamma+1)(\omega_\alpha \cdot \mathcal{L}) - (\gamma-1)(\mathcal{L} \cdot \mathcal{L}) - \frac{\gamma^2+g-2\gamma}{\gamma-1}(\omega_\alpha \cdot \omega_\alpha)\right).$$

Let Γ be a general fibre of α. The last formula can be translated in terms of intersection numbers as follows

$$\frac{1}{g-\gamma}\left(2(\mathcal{L} \cdot \Gamma)(\omega_\alpha \cdot \mathcal{L}) - (\omega_\alpha \cdot \Gamma)(\mathcal{L} \cdot \mathcal{L}) - 4\frac{\gamma^2+g-2\gamma}{(\omega_\alpha \cdot \Gamma)}(\omega_\alpha \cdot \omega_\alpha)\right).$$

As $(\omega_\alpha \cdot \omega_\alpha) \geq 0$, the intersection matrix of ω_α, $\mathcal{L}$ and Γ

$$M = \begin{pmatrix} (\omega_\alpha \cdot \omega_\alpha) & (\omega_\alpha \cdot \mathcal{L}) & (\omega_\alpha \cdot \Gamma) \\ (\omega_\alpha \cdot \mathcal{L}) & (\mathcal{L} \cdot \mathcal{L}) & (\mathcal{L} \cdot \Gamma) \\ (\omega_\alpha \cdot \Gamma) & (\mathcal{L} \cdot \Gamma) & 0 \end{pmatrix}$$

cannot be negative definite. The Index Theorem then implies that its determinant is non-negative, i.e., that

$$2(\mathcal{L} \cdot \Gamma)(\omega_\alpha \cdot \Gamma)(\omega_\alpha \cdot \mathcal{L}) - (\omega_\alpha \cdot \Gamma)^2(\mathcal{L} \cdot \mathcal{L}) \geq (\mathcal{L} \cdot \Gamma)^2(\omega_\alpha \cdot \omega_\alpha).$$

Combining this inequality with the ones obtained above, we get

$$(\omega_{f'} \cdot \omega_{f'}) - 4\frac{g-1}{g-\gamma} \deg {f'}_*\omega_{f'} \geq \frac{1}{g-\gamma}\left(\frac{(\mathcal{L} \cdot \Gamma)^2}{(\omega_\alpha \cdot \Gamma)} - 4\frac{\gamma^2+g-2\gamma}{(\omega_\alpha \cdot \Gamma)}\right)(\omega_\alpha \cdot \omega_\alpha),$$

and so

$$(\omega_{f'} \cdot \omega_{f'}) - 4\frac{g-1}{g-\gamma} \deg {f'}_* \omega_{f'} \geq \frac{(g-4\gamma-1)(g-1)}{2(g-\gamma)(\gamma-1)} (\omega_\alpha \cdot \omega_\alpha). \tag{2.13}$$

The expression on the right is clearly non-negative as soon as $g \geq 4\gamma + 1$.

To prove the characterisation of the fibrations that reach the bound, observe first that the coefficient of $(\omega_\alpha \cdot \omega_\alpha)$ in (2.13) is not zero when $g > 4\gamma+1$, so the local triviality of α is a necessary condition. Then recall that $\mathcal{O}(R) = \mathcal{L}^2$ and notice that, if (2.12) is an equality, all the inequalities in the proof must be equalities, and the terms $2\sum_{i=1}^k \left(\left[\frac{m_i}{2}\right] - 1\right)\left(\frac{\gamma-1}{g-\gamma}\left[\frac{m_i}{2}\right] + 1\right)$ and ϵ in (2.11) must vanish. In particular, we get that

$$(g - 2\gamma + 1)(\omega_\alpha \cdot \mathcal{L}) - (\gamma - 1)(\mathcal{L} \cdot \mathcal{L}) = 0 \tag{2.14}$$

which, in view of of $(\omega_\alpha \cdot \omega_\alpha) = 0$ and of $\mathcal{O}(R) = \mathcal{L}^2$, is equivalent to the vanishing of the determinant of the intersection matrix of ω_α, Γ and R, i.e., to R being numerically equivalent to a linear combination of ω_α and Γ. □

The analogous result for $g = 4\gamma + 1$ can be stated as follows.

Theorem 2.3.10. *Let $f\colon X \to B$ be a double fibration of type (g, γ) with $g = 4\gamma + 1$. Then inequality (2.12) holds, provided we are in one of the following cases:*

1. *f is a double cover fibration;*
2. *f is a semi-stable fibration.*

In particular, (2.12) is valid if a smooth fibre of f admits an involution whose quotient is a non-hyperelliptic curve of genus γ.

Moreover, a necessary condition for the slope to reach the bound is that the associated relatively minimal fibration of genus γ be either locally trivial, or hyperelliptic with slope $4(\gamma - 1)/\gamma$.

Proof. case (1) is covered by the argument used to prove Theorem 2.3.9. In case (2) we can obtain a double cover fibration after a base change, and the slope remains unchanged by Theorem 2.1.1; so again we can apply the argument of Theorem 2.3.9. It follows from Lemma 2.3.4 and the comment immediately following it that a sufficient condition for f to be a double cover fibration is that *one* of its smooth fibres admit an involution with non-hyperelliptic genus γ quotient. The coefficient of $(\omega_\alpha \cdot \omega_\alpha)$ in inequality (2.13) is 0 in this case. Hence the local triviality of α is no more a necessary condition for the fibrations to reach the bound. If α is not locally trivial, it is instead necessary that α itself attain the bound given by the slope inequality, so we conclude. □

Clearly, one could give necessary and sufficient conditions as in Theorem 2.3.9, imposing that the inequalities in the proof be equalities. It is interesting to notice that, in this borderline case, the conditions change substantially, because local triviality of the fibration of genus γ is no more needed, and indeed one can construct a fibred surface of arbitrary genus g reaching the bound and which is a double cover of a non locally trivial fibration of genus $\gamma = (g-1)/4$ (Example 2.3.12).

2.3.5 Examples

We present below two examples, both due to Barja (cf. [Bar], sec. 4.5) that show that the bound given is indeed sharp. The first is an example of double cover fibrations reaching the bound; in the second we construct a fibration with $g = 4\gamma + 1$, reaching the bound, which is a double cover of a hyperelliptic fibration which in turn reaches the bound given by the slope inequality. A more general version of the same construction leads to counterexamples to the bound for $g < 4\gamma$.

Example 2.3.11. This is a generalisation of the examples of hyperelliptic fibrations reaching bound of the slope inequality constructed in [Xia87b] and in [CH88].

Let Γ and B be smooth curves. Call γ the genus of Γ. Let $p_1 \colon B\times\Gamma \to B$ and $p_2 \colon B\times\Gamma \to \Gamma$ be the two projections, and H_1, H_2 their general fibres. For sufficiently large integers n and m, the linear system $|2nH_1 + 2mH_2|$ is base-point-free. Hence, by Bertini's Theorem there exists a smooth divisor $R \in |2nH_1 + 2mH_2|$. As R is even, we can construct the double cover $\rho\colon X \to B \times \Gamma$ ramified over R.

$$\begin{array}{ccc} & X & \\ & \rho \downarrow & \\ & B\times\Gamma & \\ {}^{p_1}\swarrow & & \searrow{}^{p_2} \\ B & & \Gamma \end{array}$$

Consider the fibration $f := p_1 \circ \rho\colon X \to B$; its general fibre is a double cover of Γ, and its genus is $g = 2\gamma + m - 1$. Observe that

$$\omega_f \sim \rho^*(\omega_{p_1}(nH_1 + mH_2)) \equiv \rho^*(nH_1 + (2\gamma - 2 + m)H_2),$$

and

$$\deg f_*\omega_f = \deg p_{1*}(\omega_{p_1}(nH_1 + mH_2)) = n(\gamma - 1 + m).$$

Therefore slope of f is exactly

$$s(f) = 4\frac{2\gamma + m - 2}{\gamma + m - 1} = 4\frac{g-1}{g-\gamma}.$$

If we consider a general divisor $R \in |2nH_1 + 2mH_2|$, it has only simple ramification points over B, and we obtain a semi-stable fibration.

Note that R is numerically equivalent to a linear combination of ω_{p_1} and Γ, as it should be, because

$$R \equiv 2nH_1 + 2mH_2 \equiv 2n\Gamma + \frac{m}{\gamma - 1}\omega_{p_1}.$$

Therefore, also condition (2.14) is satisfied; indeed,

$$(\omega_{p_1} \cdot \mathcal{L}) = (p_2^*\omega_\Gamma \cdot nH_1 + mH_2) = (2\gamma - 2)n,$$

and

$$(\mathcal{L} \cdot \mathcal{L}) = (nH_1 + mH_2 \cdot nH_1 + mH_2) = 2mn.$$

Clearly, choosing $\gamma = 0$, we obtain hyperelliptic fibrations of arbitrary genus whose slope reaches the bound $4(g-1)/g$.

Example 2.3.12. Consider the fibration of Example 2.3.11 with $\Gamma = \mathbb{P}^1$, and set $Z = X$, $f_1 = p_1 \circ \rho$, $f_2 = p_2 \circ \rho$. Call F_i the general fibre of f_i; hence F_1 is hyperelliptic of genus $\gamma := g(F_1) = m - 1$. By what we observed in Example 2.3.11,

$$\omega_{f_1} \sim \rho^*(\omega_{p_1}(nH_1 + mH_2)) \equiv nF_1 + (m-2)F_2$$

Let x, y be positive integers, and consider the linear system $|2xF_1 + 2yF_2|$. Applying Bertini's theorem again, for large enough x and y we can find a smooth even divisor Δ belonging to it. Call $\pi\colon X \to Z$ the double cover ramified over Δ. Call f the fibration $p_1 \circ \rho \circ \pi\colon X \to B$.

$$\begin{array}{ccccc} & & X & & \\ & & \pi \downarrow & & \\ & & Z & & \\ & & \rho \downarrow & & \\ & & B \times \mathbb{P}^1 & & \\ & {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} & \\ B & & & & \mathbb{P}^1 \end{array}$$

The general fibre F of f is a double cover of F_1. So f is a double cover fibration over an hyperelliptic fibration with minimal slope $4(\gamma - 1)/\gamma$. Its genus is

$$g = 2(m-1) + 2y - 1.$$

Now, $\omega_f \sim \pi^*(\omega_{f_1}(xF_1 + yF_2) \equiv (n+x)F + (m+y-2)\pi^*F_2$, so

$$(\omega_f \cdot \omega_f) = 8(x+n)(y+m-2),$$

while, using the theory of cyclic coverings,

$$f_*\omega_f = f_{1*}(\omega_{f_1}(xF_1 + yF_2)) \oplus f_{1*}(\omega_{f_1}) = p_{1*}(\rho_*(\rho^*((n+x)H_1 + (m+y-2)H_2))) =$$

$$= p_{1*}(\omega_{p_1}((n+x)H_1 + (m+y)H_2)) \oplus p_{1*}(xH_1 + (y-2)H_2) \oplus f_{1*}(\omega_{f_1}).$$

Therefore

$$\deg f_*\omega_f = (x+n)(y+m-1) + x(y-1) + n(m-1).$$

If we choose, as we may, $m = y$, we get $g = 4\gamma + 1$, and the slope becomes

$$s(f) = 8\frac{2m-2}{3m-2} = 4\frac{g-1}{g-\gamma}.$$

Observe moreover that $s(f) < 4(g-1)/(g-\gamma)$ if and only if $m > y$. In this case $g \leq 4\gamma - 1$. So this example provides also counterexamples for these cases.

2.4 The slope of non-Albanese fibrations

Recall from the first section that that $f : X \to B$ is a non-Albanese fibration if and only if $q - b > 0$, where q is the irregularity of X and b is the genus of the base curve B. We will call the integer $q - b$ *relative irregularity* of f and indicate it with the symbol q_f.

In this section we partially prove, applying the Cornalba-Harris Theorem, a result of Xiao. This result has to be regarded as a first step in the application of the Cornalba-Harris method to this setting.

We shall use a result of Fujita ([Fuj78]) that provides a decomposition of the sheaf $f_*\omega_f$. This will allow us to find natural subsheaves of $f_*\omega_f$, to which we can apply Theorem 1.5.1.

Let us first give some definitions. Let $\mathcal{F}$ be a locally free sheaf on B, and let

$$\mathbb{P}_B(\mathcal{F}) \to B$$

be the associated projective bundle. The sheaf $\mathcal{F}$ is said to be *nef* (resp. *ample*) if the tautological bundle $\mathcal{O}_{\mathbb{P}_B(\mathcal{F})}(1)$ is nef (resp. ample). The property we shall use is that if $\mathcal{F}$ is nef then its quotients have non-negative degree. In other words, any subsheaf of $\mathcal{F}$ has degree smaller or equal to $\deg \mathcal{F}$ (cf. [Vie95]).

We will use the following results.

Theorem 2.4.1. *Let $f : X \to B$ be a relatively minimal fibred surface.*

- *(Fujita, [Fuj78]) The vector bundle $f_*\omega_f$ has a decomposition*

$$f_*\omega_f = \mathcal{A} \oplus \mathcal{O}_B^{q_f},$$

where $\mathcal{A}$ is a nef vector bundle.

- *(Viehweg, [Vie95]) The sheaf $f_*\omega_f^h$ is nef for $h \geq 1$. Moreover, if f is non isotrivial, $f_*\omega_f^h$ is ample for $h \geq 2$.*

2.4.1 A new proof of a result of Xiao

In the case $q_f \geq 2$, we can prove, via the Cornalba-Harris method, that 4 is a lower bound on the slope for non-hyperelliptic and non-trigonal fibrations. This was first proved by Xiao in [Xia87a] (corollary 1 pag.459). Xiao's result, however, holds also for hyperelliptic and trigonal fibrations.[2] See also [Bar] Theorem 4.19 for a different proof of Xiao's bound.

Theorem 2.4.2. *Let $f : X \to B$ be a non-hyperelliptic and non-trigonal fibration of genus $g \geq 5$ such that the general fibres of f are not smooth plane quintics. Suppose that $q_f \geq 2$. Then*

$$s(f) \geq 4.$$

[2] In particular we can deduce from it that the hyperelliptic fibrations with slope $4(g-1)/g$ are Albanese fibrations. For the examples constructed in Example 2.3.11, this can be easily proven directly (see Remark 2.4.4).

Proof. Consider Fujita's decomposition $f_*\omega_f = \mathcal{A} \oplus \mathcal{O}_B^{q_f}$. Write $\mathcal{O}_B^{q_f} = \oplus_{i=1}^{q_f} \mathcal{O}_B \phi_i$, where the ϕ_i are relative differentials, and denote by φ_i the restriction of ϕ_i to X_t. Let $\mathcal{E}$ be the sheaf $\mathcal{A} \oplus (\oplus_{i=3}^{q_f} \mathcal{O}_B \phi_i)$, and consider the corresponding decomposition on a general fibre X_t

$$H^0(X_t, K_{X_t}) = E \oplus W,$$

where $E = \mathcal{E} \otimes \boldsymbol{k}(t)$ and $W = (\mathcal{O}_B\phi_1 \oplus \mathcal{O}_B\phi_2) \otimes \boldsymbol{k}(t) = \langle \varphi_1, \varphi_2 \rangle$. The linear system E induces the projection of the canonical image of X_t from the line $\mathbb{P}(Ann(E)) \subset \mathbb{P}(H^0(X_t, K_{X_t})^\vee)$. Let $\{P_1, \dots P_n\} = \mathbb{P}(Ann(E)) \cap X_t$ be the base locus of this projection.

The P_i's are base points for the differentials belonging to the linear subsystem $E \subset H^0(X_t, K_{X_t})$, hence they are not base points for W. The morphisms

$$\psi_i : \mathbb{C}^2 \longrightarrow \omega_f \otimes \boldsymbol{k}(P_i) \cong \mathbb{C}$$

defined by $\psi_i(x_1, x_2) = x_1\varphi_1(P_i) + x_2\varphi_2(P_i)$, are therefore surjective. Hence, for any couple $(a_1, a_2) \in \mathbb{C}^2 \setminus (\cup_i \ker(\psi_i))$, the linear combination $a_1\varphi_1 + a_2\varphi_2 \in W$ does not vanish in *any* of the P_i's. Let $\mathcal{H} \subseteq \mathcal{O}_B^{q_f}$ be the (of course trivial) sheaf generated by $a_1\phi_1 + a_2\phi_2$; by construction, the fibre of $\mathcal{E} \oplus \mathcal{H}$ at general t is a base-point free linear subsystem of the canonical system. In other words, we have chosen a point outside X_t on the line $\mathbb{P}(Ann(E))$ for t general.

Consider the sheaf $\mathcal{F} = \mathcal{E} \oplus \mathcal{H}$. Let π_t be the morphism associated to $\mathcal{F} \otimes \boldsymbol{k}(t)$ at general X_t. By construction, π_t is the projection from a point in $\mathbb{P}^{g-1}$ *disjoint* from the canonical image of X_t. By Proposition 1.3.17 π_t is birational. By Theorem 1.3.8 and Theorem 1.4.2 it is a Hilbert stable morphism. We may therefore apply Corollary 1.5.4 of Theorem 1.5.1. Consider the homomorphism

$$\mathrm{Sym}^h \mathcal{F} \longrightarrow f_*\omega_f^h,$$

and call $\mathcal{G}_h$ its image. As π_t is birational of degree $2g-2$, we conclude that

$$\mathrm{rank}\mathcal{G}_h = h^0(\overline{X}_t, (j^*\mathcal{O}_{\mathbb{P}^{g-2}}(1))^h) = (2g-2)h + O(1),$$

where $\overline{X}_t$ is the image of X_t. Moreover,

$$\deg \mathcal{G}_h \le \deg f_*\omega_f^h$$

because $f_*\omega_f^h$ is nef. Hence, the coefficient of h^2 in $\deg \mathcal{G}_h$ is smaller than $(\omega_f \cdot \omega_f)/2$, and inequality (1.12) implies

$$(g-1)\frac{(\omega_f \cdot \omega_f)}{2} - (2g-2)\deg f_*\omega_f \ge 0,$$

as claimed. □

In the case of trigonal fibrations we can state the following result.

Proposition 2.4.3. *Let $f\colon X \to B$ be a trigonal fibration of genus $g \ge 5$. Suppose that $q_f \ge 3$. Then*

$$s(f) \ge 4.$$

Proof. Consider the sheaf $\mathcal{E} = \mathcal{A} \oplus \mathcal{O}_B^{q_f-3}$. Its fibre E at general t induces the projection of the canonical image of X_t from the plane $\Pi = \mathbb{P}(Ann(E)) \subset \mathbb{P}^{g-1}$ We can choose a point P in Π disjoint from the image of X_t and not lying in any trisecant line. Indeed, the variety of trisecant lines has dimension 2 (and of course it is not a plane). So a general point of Π satisfies our conditions. Let E' be the linear system corresponding to the projection from P. Notice that we can extend E' to a trivial direct factor $\cong \mathcal{O}_B$ of $f_*\omega_f$, getting a decompostion

$$f_*\omega_f = \mathcal{E}' \oplus \mathcal{O}_B.$$

By construction, the fibre of $\mathcal{E}'$ at general t induces a projection satisfying the assumption of Proposition 1.3.10. Hence we can apply the Cornalba-Harris Theorem to $\mathcal{E}'$. Arguing as in the last part of the proof of the above theorem, we conclude the proof. □

Note that for trigonal fibrations Konno's bound $s(f) \geq 14(g-1)/(3g+1)$ implies that 4 is a lower bound for the slope as soon as $g \geq 9$, independently of the relative irregularity.

A conjecture on a sharper bound

In order to obtain better results via the Cornalba-Harris method, it is clear that we need a better understanding of Fujita's decomposition, and in particular the geometry of the linear systems induced by it on the general fibres. We now make some reflections, as a guideline for further applications.

Suppose that, under suitable assumptions, the fibre of $\mathcal{A}$ itself on general $b \in B$ was a base point free linear system of degree d which induced a Hilbert semi-stable morphism. Of course in this case it has to be $q_f \leq g-2$, otherwise $\mathcal{A}$ would induce a contraction on general fibres. The Cornalba-Harris Theorem would give as result the inequality

$$s(f) \geq 2\frac{d}{g-q_f}. \tag{2.15}$$

Of course $d \leq 2g-2$, and by Clifford's theorem $d \geq 2g-2q_f$. If we assume in addition that the general fibre of f has Clifford index greater or equal than q_f, then $d \geq 2g-2-q_f$. Indeed, if this were not the case, then the degree d of the projection would be smaller than $2g-2-q_f$. Consider a divisor D belonging to the linear system $\mathcal{A} \otimes \boldsymbol{k}(t)$. By Riemann-Roch

$$h^1(D) = h^0(D) + g - 1 - d \geq 2g - 1 - q_f - d > 1,$$

so D would contribute to the Clifford index of X_t, but $\mathrm{Cliff}(D) < q_f$, contrary to our assumption. Hence in this case inequality (2.15) would become

$$s(f) \geq 2\frac{2g-2-q_f}{g-q_f}$$

If $d = 2g-2$, without any assumption on q_f, we would obtain the suggestive bound

$$s(f) \geq 4\frac{g-1}{g-q_f}, \quad \text{for } q_f \leq g-2. \tag{2.16}$$

Note that this bound is perfectly analogous to the bound for double fibrations.

Remark 2.4.4. The examples of double fibrations of type (g,γ) reaching the bound given in section 2.3, have relative irregularity $q_f = \gamma$. Indeed, recall that we have constructed them as double covers of a trivial fibration

$$X \xrightarrow{\rho} B \times \Gamma \xrightarrow{p_1} B,$$

associated to a line bundle $\mathcal{L}$ numerically equivalent to $nH_1 + mH_2$ (where H_1 and H_2 are the general fibres of the projections on the factors of $B \times \Gamma$). Hence,

$$q = h^1(B\times\Gamma, \mathcal{O}_{B\times\Gamma}) + h^1(B\times\Gamma, \mathcal{L}^{-1}) = b+\gamma+h^1(B, K_B(nP_1)) + h^1(\Gamma, K_\Gamma(mP_2)) = b+\gamma.$$

Hence, the double fibrations of the example reaching the bound have exactly slope (2.16). Notice that the gonality of the general fibre of these fibrations is smaller or equal to twice the gonality of the quotient Γ, and so it is smaller or equal to $\gamma+3$. Hence, the Clifford index of the geneal fibre X_t is smaller or equal to γ. If we choose the curve Γ with maximal Clifford index, and genus γ odd, then X_t has Clifford index precisely γ.

Note that the bound (2.16) for $q_f = 1$ coincides with Xiao's result. Keeping in mind the above considerations, it seems sensible to conjecture the bound (2.16) to hold for fibrations with $\mathrm{Cliff}(f) \geq q_f$.

We mention here another reason why *a posteriori* this bound seems very reasonable, at least for semi-stable fibrations. Let $f : X \to B$ be a semi-stable fibration with s singular fibres. Vojta proved in [Lan88] (Appendix ♮ 2, see also [Tan95]) the following "canonical class inequality".

$$(\omega_f \cdot \omega_f) \leq (2g-2)(2b-2+s).$$

Using Noether's formula this inequality becomes

$$\deg f_*\omega_f \leq \frac{g-1}{6}(2b-2+s) + \frac{e_f}{12}.$$

If we now combine it with the slope inequality, as Vojta himself observes, we obtain the following bound on the degree of $f_*\omega_f$

$$\deg f_*\omega_f \leq \frac{g}{2}(2b-2+s).$$

However, a sharper bound of this type holds, (cf. [Ara71] and [VZ]), namely

$$\deg f_*\omega_f \leq \frac{g-q_f}{2}(2b-2+s). \tag{2.17}$$

If we use inequality (2.16) instead of the slope inequality in the above discussion, we obtain exactly (2.17).

The problem of an upper bound for q_f

An interesting problem related to the above bounds is the one of determining upper limitations for the relative irregularity depending on the genus of f. Xiao has proven in [Xia87a] that

$$q_f \leq \frac{1}{6}(5g+1),$$

and in [Xia87b] that

$$q \le \frac{1}{2}(g+1);$$

and he has conjectured this last bound to hold in general for the relative irregularity:

$$q_f \le \frac{1}{2}(g+1).$$

However, Pirola in [Pir92] has constructed a counterexample to this conjecture. More precisely, he shows that there exist fibred surfaces with $q_f = 3$ and genus 4.

If the bound (2.16) did hold, combining it with the upper bound on the slope $s(f) \le 12$, we would obtain

$$q_f \le \frac{2g+1}{3}, \text{ for } q_f \le g-2.$$

Bibliography

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. vol.II*, in preparation.

[ACGH85] ———, *Geometry of algebraic curves. vol.I*, Springer-Verlag, 1985.

[AK00] T. Ashikaga and K. Konno, *Global and local properties of pencils of algebraic curves.*, Adv. Stud. in Pure Math. **2** (2000), 1–49.

[Ara71] S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Math. U.S.S.R. Izv. **5** (1971), 1277–1302.

[Bad00] L. Badescu, *Algebraic Surfaces*, Universitext, Springer- Verlag, 2000.

[Bal86] E. Ballico, *On the Clifford index of algebraic curves*, Proc. of the AMS **97** (1986), no. 2, 217–218.

[Bar] M. A. Barja, *On the slope and geography of fibred surfaces and threefolds*, Ph. D. Thesis, Univesity of Barcelona, 1998.

[Bar01] ———, *On the slope of bielliptic fibrations*, Proc. Amer. Math. Soc. **129** (2001), no. 7, 1899–1906.

[Bea78] A. Beauville, *Surfaces algébriques complexes*, Astérisque, vol. 54, Société Mathématique de France, 1978.

[Bea82] ———, *L'inégalité $p_g \geq 2q + 4$ pour les surfaces de type général*, Bull. Soc. Math. Franc. **110** (1982), 343–346.

[BFM79] P. Baum, W. Fulton, and R. MacPherson, *Riemann-Roch and topological k-theory for singular varieties*, Acta Math. **143** (1979), no. 3-4, 155–192.

[BHPdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact Complex Surfaces, second edition*, Springer-Verlag, 2004.

[BN98] M. A. Barja and J. C. Naranjo, *Extension of maps defined on many fibres. Dedicated to the memory of Fernando Serrano*, Collect. Math **49** (1998), no. 2-3, 227–238.

[Bos94] J. B. Bost, *Semi-stability and heights of cycles*, Invent. Math. **118** (1994), 223–253.

[BZ01] M. A. Barja and F. Zucconi, *On the slope of fibred surfaces*, Nagoya Math. J. **164** (2001), 103–131.

[CH88] M. Cornalba and J. Harris, *Divisor classes associated to families of stable varieties, with application to the moduli space of curves*, Ann. Sc. Ec. Norm. Sup. **21** (1988), 455–475.

[CK90] M. Coppens and T. Kato, *The gonality of smooth curves with plane models*, Manusc. Math. **70** (1990), 5–25.

[CM91] M. Coppens and G. Martens, *Secant spaces and Clifford's theorem*, Comp. Math. **78** (1991), 193–212.

[Cop91] M. Coppens, *The gonality of general smooth curves with a prescribed plane nodal model*, Math. Ann. **289** (1991), 89–93.

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext., Springer-Verlag, 2001.

[EH87] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90** (1987), no. 2, 359–387.

[Fuj78] T. Fujita, *On Kaeler fibre spaces over curves*, J. Math. Soc. Japan **30** (1978), no. 4, 779–794.

[Ful84] W. Fulton, *Intersection theory*, Erg. der Math., vol. 3, Springer-Verlag, 1984.

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley & Sons, 1978.

[Gie77] D. Gieseker, *Global Moduli for Surfaces of General Type*, Invent. Math. **43** (1977), 233–282.

[Gie82] ———, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, Springer- Verlag, 1982.

[Har77] R. Hartshorne, *Algebraic Geometry*, G.T.M., vol. 52, Springer-Verlag, 1977.

[Har84] J. Harris, *On the Kodaira dimension of the moduli space of curves. II. The even-genus case*, Invent. Math. **75** (1984), no. 3, 437–466.

[Har02] R. Hartshorne, *Clifford index of ACM curves in* $\mathbb{P}^3$, Milan J. Math. **70** (2002), 209–221.

[HM82] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, With an appendix by William Fulton.

[HM98] J. Harris and I. Morrison, *Moduli of Curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, 1998.

[Hor81] E. Horikawa, *Algebraic surfaces of general type with small* c_1^2*, V*, Fac. Sci. Univ. Tokyo, Sect IA **283** (1981), 745–755.

[Jac85] N. Jacobson, *Basic Algebra I*, W. H. Freemann and Company, 1985.

[Kem78] G. R. Kempf, *Instability in invariant theory*, Ann. of Math. **108** (1978), 299–316.

[KMM87] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the Minimal Model Problem*, Adv. Stud. in Pure Math. **10** (1987), 283–360.

[Kon93] K. Konno, *Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces*, Ann. Sc. Norm. Sup. Pisa ser. IV **20** (1993), 575–595.

[Kon94] ______, *On the Irregularity of Special Non-Canonical Surfaces*, Publ. RIMS Kyoto Univ. **30** (1994), 671–688.

[Kon96] ______, *A lower bound of the slope of trigonal fibrations*, Internat. J. Math. **7** (1996), no. 1, 19–27.

[Kon99] ______, *Clifford index and the slope of fibered surfaces*, J. Algebraic Geom. **8** (1999), no. 2, 207–220.

[Lan88] S. Lang, *Introduction to Arakelov Theory*, Springer-Verlag, 1988.

[Mar67] H. H. Martens, *On the varieties of special divisors on a curve. I*, J. reine angew. Math. **227** (1967), 111–120.

[Mar68] ______, *On the varieties of special divisors on a curve. II*, J. reine angew. Math. **233** (1968), 89–100.

[Mar80] G. Martens, *Funktionen von vorgegebener Ordnung auf komplexen Kurven*, J. reine angew. Math. **320** (1980), 68–85.

[Mat90] S. Matsusaka, *Some numerical invariants of hyperelliptic fibrations*, J. Math. Kyoto Univ. **30-1** (1990), 33–57.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory (third edition)*, Erg. Math., Springer-Verlag, 1994.

[Mir95] R. Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics, vol. 5, AMS, 1995.

[Mor80] I. Morrison, *Projective Stability of Ruled Surfaces*, Invent. Math. **56** (1980), 269–304.

[Mor96] A. Moriwaki, *A sharp slope inequality for general stable fibrations of curves*, J. Reine Angew. Math. **480** (1996), 177–195.

[Mor02] ______, *Nef divisors in codimension one on the moduli space of stable curves*, Compositio Math. **132** (2002), no. 2, 191–228.

[Mum74] D. Mumford, *Geomtric Invariant Theory*, Erg. Math., Springer-Verlag, 1974.

[Mum77] ______, *Stability of projective varieties*, L'Ens. Math. **23** (1977), 39–110.

[Nam79] M. Namba, *Families of Meromorphic Functions on Compact Riemann Surfaces*, Lecture Notes in Mathematics, vol. 767, Springer-Verlag, 1979.

[Per82] U. Persson, *Chern invariants of surfaces of general type*, Composito Math. **43** (1982), 3–58.

[Pir92] G. P. Pirola, *On a conjecture of Xiao*, J. Reine Angew. Math. **431** (1992), 75–89.

[Rei88] M. Reid, *Quadrics through a canonical surface*, Algebraic Geometry (L'Aquila) Lecture notes in Math. **1147** (1988), 191–213.

[Ser96] F. Serrano, *Isotrivial Fibred Surfaces*, Ann. Mat. Pura Appl. **171** (1996), 63–81.

[SF00] Z. E Stankova-Frenkel, *Moduli of trigonal curves*, J. Algebraic Geom. **9** (2000), no. 4, 607–662.

[Tan94] S. Tan, *On the invariants of base changes of pencils of curves I*, Manuscripta Math. **84** (1994), 225–244.

[Tan95] ———, *The minimal number of singular fibres of a semistable curve over* $\mathbb{P}^1$, J. Algebraic Geometry **4** (1995), 591–596.

[Tan96] ———, *On the invariants of base changes of pencils of curves II*, Math. Z. **222** (1996), 655–676.

[Vie95] E. Viehveg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 30, Springer-Verlag, 1995.

[VZ] E. Viehweg and K. Zuo, *Numerical bounds for semi-stable families of curves or of certain higher dimensional manifolds*, Preprint 2005, math.AG/0504542.

[Xia87a] G. Xiao, *Fibred algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466.

[Xia87b] ———, *Irregularity of surfaces with a linear pencil*, Duke Math. J. **55** (1987), no. 3, 597–602.